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BEHAVIOR OF SOLUTIONS TO GENERALIZED DIFFERENTIAL EQUATIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "BEHAVIOR OF SOLUTIONS TO GENERALIZED DIFFERENTIAL EQUATIONS" submitted by JOHN W. VAN KIRK in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This paper considers a problem of the form

$$(P)\quad x'(t) \in F(t,x(t))$$
$$x(t_o) = x_o$$

where F is a set-valued mapping from R^{n+1} into subsets of R^n. The basic hypotheses on F require that it be continuous in x, measurable in t, and that $|y| \leq m(t)$ for all $y \in F(\cdot,t)$ and for some integrable function m. In addition, it will be assumed that F maps into compact, convex subsets of R^n. Under these assumptions, a global existence theorem for (P) will be proved. In addition, if B is a compact, connected set in R^n and if A is the set of all points in R^n which can be reached by solutions to (P) with $x(t_o) \in B$, then it is shown that A is also compact and connected under the above hypotheses.

In order to prove these theorems, some basic theory of set-valued functions is developed. The concepts of measurability and continuity are defined, as well as the concept of an integral of set-valued functions. Some fundamental results of the theory of real variables are then generalized.

In the last chapter, the behavior of solutions relative to subsets V of R^{n+1} is considered. First, conditions are imposed on two surfaces $x_1 = z(t,x_2,\cdots,x_n)$ and $x_1 = w(t,x_2,\cdots,x_n)$ such that if $S \subset \{t_o\} \times R^n$ is a connected subset intersecting both surfaces then for every time $t \geq t_o$,

there is some solution x to (P) with $(t_o, x(t_o)) \in S$ and $(t, x(t)) \in V$.

Second, Wazewski's theorem [14, p. 280] is generalized. In this theorem, conditions are imposed on $S \subset V$ and V which insure that at least one solution x with $x(t_o) \in S$ remains in V for all $t \geq t_o$. The conditions require that if a solution leaves V , it leaves for some finite time before returning to V . Using this theorem and strengthening the hypotheses on the previous problem with the two surfaces, it is shown that some solution x with $(t_o, x(t_o)) \in S$ must remain between the two surfaces for all time $t \geq t_o$.

Finally, it is shown that if a function v satisfies certain hypotheses and if $x(t_o) \in \{x : v(x) \leq \lambda\}$ for some solution to (P) and any λ , then $x(t) \in \{x : v(x) \leq \lambda\}$ for $t \geq t_o$.

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CHAPTER I

HISTORICAL BACKGROUND

The study of contingent and paratingent equations was begun by Marchaud [23,24] and Zaremba [34,35] in the middle 1930's. The basic problem in both theories was to determine the behavior of solutions to first order differential equations whose right hand side was not uniquely determined.

To be more specific, let us consider the formulation of their problem. Marchaud's contingent theory considered half-tangents to a curve. A ray L with tip x belongs to the set of right half-tangents to a curve C at a point x, which is an accumulation point of C, if there exists a sequence of points x_n "to the right of x on C" such that x_n tends to x and the line segment xx_n tends to a segment of L. $C(x)$ is an admissable vector field if it consists of convex half-cones with tip x all of which make an angle at most equal to some fixed angle less than $\frac{\pi}{2}$. Also, $C(x)$ must be a continuous function in the Hausdorff metric (see remarks after Lemma 2.6). An integral of $C(x)$ is a simple arc such that all right half-tangents from an accumulation point x of the arc belong to $C(x)$ and all left hand-tangents from x belong to $-C(x)$.

Zaremba, on the other hand, considered the paratingent of a curve. A line L belongs to the paratingent of a curve C at an accumulation point x of C if there exist two sequences x_n and y_n tending to x and belonging to C such that the line segment $x_n y_n$ tends to a segment

of L. $N(t,x)$ is an admissable vector field if it consists of a pencil of straight lines with tip x, i.e. a continuum of straight lines which pass through a common point called its tip and such that any plane passing through two lines in the pencil cuts the pencil in a continuum. Also $N(t,x)$ must be an upper semicontinuous function (see Definition 2.4). An integral of $N(t,x)$ is any Jordan curve or simple arc C, parametrized by t, such that at every accumulation point x_t of C, the paratingent of C at x_t is contained in the pencil $N(t,x_t)$.

In [35], Zaremba showed that his theory was just a generalization of that of Marchaud. Many of the basic properties of solutions were developed in these papers. For the next 25 years, little was done in this area. Some work was done by Hukuhara [18] on differential inequalities, a special case of the above theory.

Interest was rekindled around 1960 by some work of Wazewski [30,31]. He proved two important theorems. First, he showed that the paratingent problem associated with the pencil $N(t,x)$ was equivalent to finding an absolutely continuous function $x(t)$ such that $x'(t) \in N^*(t,x(t))$ a.e. where $N^*(t,x)$ is the set of slopes of the straight lines in $N(t,x)$. Second, he showed that the control problem $x'(t) = f(t,x(t),u(t))$, u measurable, and $u(t) \in C(t,x(t))$ a.e. is equivalent to the paratingent equation associated with $N(t,x)$, the set of all lines whose slope belongs to the sets $\bigcup_{u \in C(t,x)} \{a : a = f(t,x,y)\}$. The first result allows a more analytical approach to be taken and the second gives an important application of the theory which was previously lacking. We will return to this in chapter 3.

CHAPTER II

THEORY OF SET-VALUED FUNCTIONS

Before considering the basic problem, some theorems on set-valued functions are required. First, some notation is necessary.

Definition 2.1: The space of all nonempty, compact subsets of R^n is denoted by $c(R^n)$. The space of all nonempty, compact, and convex subsets of R^n is denoted by $cc(R^n)$.

Definition 2.2: Let $d(x,y)$ be the Euclidean metric on R^n and let $A,B \in c(R^n)$, then

(a) $d_1(x,A) = \inf\{d(x,y) : y \in A\}$

(b) $D_1(A,B) = \sup\{d_1(x,B) : x \in A\}$

(c) $D(A,B) = \max\{D_1(A,B) , D_1(B,A)\}$

(d) $\mathrm{dist}(A,B) = \inf\{d(x,y) : x \in A , y \in B\}$.

Definition 2.3: $S(A,\epsilon) = \{x : d_1(x,A) < \epsilon\}$.

Definition 2.4: Let E be a subset in R^n and $F : E \to c(R^n)$ be a function; then

(a) F is lower semicontinuous (l.s.c.) at a point a in E , if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$ with $d(x,a) < \delta$, $F(a) \subset S(F(x),\epsilon)$.

(b) F is upper semicontinuous (u.s.c.) at a point a in E if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$ with $d(x,a) < \delta$, $F(x) \subset S(F(a),\epsilon)$.

(c) F is continuous at a in E if it is u.s.c. at a and l.s.c. at a.

(d) F is measurable on E if $\{x \in E : F(x) \cap C \neq \phi\}$ is measurable for every $C \in c(R^n)$.

Remark: $m(E)$ denotes the Lebesque measure of E.

Lemma 2.5: $d_1(x,A) \leq d(x,y) + d_1(y,A)$.

Proof:

$$\begin{aligned} d_1(x,A) &= \inf\{d(x,z) : z \in A\} \\ &\leq \inf\{d(x,y) + d(y,z) : z \in A\} \\ &\leq d(x,y) + d_1(y,A) . \end{aligned}$$

Lemma 2.6: $c(R^n)$ is a metric space with the metric $D(A,B)$ defined above.

Proof: $D(A,B)$ is obviously positive and symmetric. The compactness of A and B give that $D(A,B) = 0$ implies $A = B$. Therefore only the triangle inequality need be proved.

A preliminary observation is needed first. If A and B are compact, then choose S compact such that $S \supset A \cup B$. Then $d(x,y)$ is bounded on S, so by 2.3.7 in [9],

$$\sup\{\sup\{d(x,y):x\in A\},y\in B\} = \sup\{\sup\{d(x,y),y\in B\},x\in A\} \quad .$$

With this, it is seen that

$$\begin{aligned}
D(A,B) &= \max\{\sup\{d_1(x,B):x\in A\} \ , \ \sup\{d_1(y,A):y\in B\}\} \\
&\leq \max\{\sup\{d(x,z) + d_1(z,B):x\in A\} \ , \ \sup\{d(y,z) + d_1(z,A):y\in B\}\} \\
&\leq \max\{d_1(z,A),\sup\{d(x,z):x\in A\}\} + \max\{d_1(z,B),\sup\{d(y,z):y\in B\}\} \\
&\leq \max\{\sup\{d_1(z,A):z\in C\} \ , \ \sup\{\sup\{d(x,z):z\in C\} \ , \ x\in A\}\} \\
&\quad + \max\{\sup\{d_1(z,B):z\in C\} \ , \ \sup\{\sup\{d(y,z):z\in C\} \ , \ y\in B\}\} \\
&\leq D(A,C) + D(C,B) \quad .
\end{aligned}$$

Remarks: $D(A,B)$ is called the Hausdorff metric on $c(R^n)$. From the definition, it is obvious that $F : E \to c(R^n)$ is continuous in the sense of definition 2.4.c if and only if it is continuous in the topology induced by the Hausdorff metric.

In what follows, it will be useful to have several conditions equivalent to measurability. These originate primarily with Hukuhara [19].

Lemma 2.7: Let $F : E \to c(R^n)$ where E is a bounded, measurable set. Then the following are equivalent.

(a) F is measurable .

(b) $\{x \in E : F(x) \cap B \neq \phi\}$ is measurable for each closed subset B .

(c) $\{x \in E : F(x) \subset G\}$ is measurable for each open subset G .

(d) $\{x \in E : F(x) \subset C\}$ is measurable for each compact subset C .

(e) $\{x \in E : F(x) \subset B\}$ is measurable for each closed subset B .

(f) $\{x \in E : F(x) \cap G \neq \phi\}$ is measurable for each open subset G .

(g) $\{x \in E : F(x) \cap D \neq \phi\}$ is measurable for each $D \in D^*$ where D^* is the set of all open spheres with rational center co-ordinates and rational radii.

Proof: To show

$$a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow f \Rightarrow g \Rightarrow a \quad .$$

$a \Rightarrow b$: Given B closed, define $B_n = B \cap S_n$ where S_n is the closed sphere of radius n . Then B_n is compact and

$$\{x \in E : F(x) \cap B \neq \phi\} = \bigcup_{n=1}^{\infty} \{x \in E : F(x) \cap B_n \neq \phi\} .$$

Each of these latter sets is measurable so (b) holds.

$$b \Rightarrow c : \{x \in E : F(x) \subset G\} = \{x \in E : F(x) \cap (R^n - G) = \phi\}$$

$$= E - \{x \in E : F(x) \cap (R^n - G) \neq \phi\} .$$

If G is open, $R^n - G$ is closed so the result follows.

$c \Rightarrow d$: Given C compact, define $C_n = \{x : d_1(x,C) < \frac{1}{n}\}$.
Then $\{x \in E : F(x) \subset C\} = \bigcap_{n=1}^{\infty} \{x \in E : F(x) \subset C_n\}$. Since C_n is open, d follows.

d => e : Given B closed, define B_n as above, then

$$\{x \in E : F(x) \subset B\} = \bigcup_{n=1}^{\infty} \{x \in E : F(x) \subset B_n\}$$

which proves the result.

e => f : $\{x \in E : F(x)^\circ \cap G \neq \phi\} = E - \{x \in E : F(x) \cap G = \phi\}$

$$= E - \{x \in E : F(x) \subset (R^n - G)\} \ .$$

If G is open, $R^n - G$ is closed giving the result.

f => g : Trivial.

g => a : Given C compact, choose $D_1^n, \cdots, D_{m_n}^n$ from D^* of radius $\frac{1}{n}$ such that $D_n \equiv \bigcup_{i=1}^{m_n} D_i^n \supset C$. Then $\{x \in E : F(x) \cap D_n \neq \phi\}$ is measurable and $\{x \in E : F(x) \cap C \neq \phi\} = \bigcap_{i=1}^{\infty} \{x \in E : F(x) \cap D_i \neq \phi\}$.

<u>Lemma 2.8</u>: If $F : E \to c(R^n)$ is u.s.c. then it is measurable.

<u>Proof</u>; Let G be an open set.

If $y \in \{x \in E : F(x) \subset G\}$, then $F(y) \subset G$, but $F(y)$ is closed so for some $\epsilon > 0$, $S(F(y),\epsilon) \subset G$. By the u.s.c. of F , there exists $\delta > 0$ such that

$$d(y,z) < \delta \ \text{ and } \ z \in E \ \text{ implies } \ F(z) \subset S(F(y),\epsilon) \subset G \ .$$

Hence $z \in \{x \in E : F(x) \subset G\}$, so $\{x \in E : F(x) \subset G\}$ is open, hence measurable, which proves the result by lemma 2.7.

Lemma 2.9: If $F : E \to c(R^n)$ is l.s.c. then it is measurable.

Proof: Let $C \in c(R^n)$ and let $\{x_i\}_{i=1}^{\infty} \subset \{x \in E : F(x) \subset C\}$, $x_i \to y$. Then $F(x_i) \subset C$ for all i and by the l.s.c. of F, given ϵ, $F(y) \subset S(F(x_i),\epsilon)$ for i large enough. Hence, $F(y) \subset S(C,\epsilon)$ for all $\epsilon > 0$, so $F(y) \subset C$. Therefore $y \in \{x \in E : F(x) \subset C\}$ and this set is closed, hence measurable, which by lemma 2.7 proves the result.

Corollary 2.10: If $F : E \to c(R^n)$ is continuous, then it is measurable.

Proof: A continuous function is u.s.c.

The following lemma is due to Plis [26].

Notation: In all that follows, I will denote a compact subset of R.

Lemma 2.11: Let $F : I \to c(R^n)$ be measurable and uniformly bounded, then for any $\epsilon > 0$ there exists a closed subset $E \subset I$ such that F is continuous on E and $m(I-E) < \epsilon$.

Proof: Let $S = \cup\{F(t) : t \in I\}$, then S is bounded so $\overline{S}$ is compact. Therefore for every $j = 1,2,\cdots$ there exists p_j and U_{jk}, $k = 1,\cdots,p_j$, such that $\operatorname{diam}(U_{jk}) < \frac{1}{j}$ and $S = \bigcup_{k=1}^{p_j} U_{jk}$. Let $K_{j1},\cdots,K_{jq_j}$, $q_j = 2^{p_j} - 1$, be the array of all possible sets of indices $1,\cdots,p_j$.

Define: $C'_{jm} = \{t : F(t) \subset \bigcup_{k\in K_{jm}} \overline{U}_{jk}\}$

$$B'_{jk} = \{t : F(t) \cap U_{jk} \neq \phi\}$$

$$A_{jm} = C'_{jm} \cap \{ \bigcap_{k\in K_{jm}} B'_{jk}\} .$$

By lemma 2.7, C'_{jm} and B'_{jk} are measurable, so A_{jm} is also measurable. By construction, $I = \bigcup_{k=1}^{q_j} A_{jk}$, $j = 1,2,\cdots$, since if $t \in I$ then $F(t) \neq \phi$ and $F(t) \subset S$, hence $F(t) \cap U_{jk} \neq \phi$ for some k . Let K_{jm} be the set of all k such that $F(t) \cap U_{jk} \neq \phi$, then $F(t) \subset \bigcup_{k \in K_{jm}} U_{jk}$ from which it follows that $t \in A_{jm}$.

If $s,t \in A_{jm}$, then $F(s) \subset \bigcup_{k \in K_{jm}} \overline{U}_{jk}$. Therefore, if $x \in F(s)$, $x \in \overline{U}_{jk}$ for some k and there will exist $y \in F(t) \cap U_{jk}$ so $D_1(F(t),F(s)) < \frac{1}{j}$. By symmetry, $D_1(F(s),F(t)) < \frac{1}{j}$, so $D(F(t),F(s)) < \frac{1}{j}$.

Choose r_{ij} s.t. $\sum_{i,j} r_{ij} < \epsilon$. Then by [24, p. 87, Th. 10] , for each j and k , $j = 1,2,\cdots$ and $k = 1,\cdots,q_j$, there exist closed subsets B_{jk} and C_{jk} such that $B_{jk} \subset A_{jk}$, $C_{jk} \subset I-A_{jk}$, and $m(I-(B_{jk} \cup C_{jk})) < r_{jk}$. Since $B_{jk} \cap C_{jk} = \phi$, dist $(B_{jk},C_{jk}) > \epsilon_{jk}$, some $\epsilon_{jk} > 0$. Letting $\epsilon_j = \min(\epsilon_{jk} : k = 1,\cdots,q_j)$, then dist $(B_{jk},C_{jk}) > \epsilon_j$ for all j and k .

Let $D_{jk} = I-(B_{jk} \cup C_{jk})$, and $E = I - \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{q_j} D_{jk}$. D_{jk} is open, so $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{q_j} D_{jk}$ is open, so E is closed. Since $m(D_{jk}) < r_{jk}$, then $m(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{q_j} D_{jk}) < \epsilon$ so $m(I-E) < \epsilon$.

If $t \in E - B_{jk}$, then $t \notin D_{jk}$ so $t \in B_{jk} \cup C_{jk}$ which implies $t \in C_{jk}$. Thus $E - B_{jk} \subset C_{jk}$, from which it follows that dist $(B_{jk} , E-B_{jk}) > \epsilon_j$.

Also, if $t \in E$, then given j since $I = \bigcup_{k=1}^{q_j} A_{jk}$, $t \in A_{jk_o}$ for some k_o. By construction of E, $t \notin D_{jk_o}$ so $t \in B_{jk_o} \cup C_{jk_o}$. But since $C_{jk_o} \cap A_{jk_o} = \phi$, $t \in B_{jk_o}$.

Finally the continuity of F on E can be shown. Choose $s_o \in E$ and $t_o \in E$ such that $d(s_o,t_o) < \epsilon_j$. Then $t_o \in B_{jk_o}$ for some k_o. Since $\text{dist}(B_{jk_o}, E-B_{jk_o}) > \epsilon_j$, $s_o \in B_{jk_o}$. However, $B_{jk_o} \subset A_{jk_o}$ so $s_o, t_o \in A_{jk_o}$ which implies $D(F(s_o),F(t_o)) < \frac{1}{j}$, which obviously proves the theorem.

The following generalization of Plis' lemma is due to Bridgeland [6, Th. 2.2].

<u>Theorem 2.12</u>: If a function $F : I \to c(R^n)$ is measurable, then for every $\epsilon > 0$, there exists a closed subset $E \subset I$ such that $m(I-E) < \epsilon$ and F is continuous on E.

<u>Proof</u>: Define $T_k = \{t \in I : F(t) \cap \{x : ||x|| > k\}\}$, $k = 1,2,\cdots$. Then $\bigcap_{k=1}^{\infty} T_k = \phi$ because if $t_o \in \bigcap_{k=1}^{\infty} T_k$ then $F(t_o) \cap \{x : ||x|| > k\} \neq \phi$ for all k, contradicting the compactness of $F(t_o)$. Also $T_i \subset T_j$ if $i > j$ and by lemma 2.7, each T_i is measurable. Hence,

$$\lim_{i\to\infty} m(T_i) = m(\bigcap_{i=1}^{\infty} T_i) = 0 \quad ,$$

so given $\epsilon > 0$, there exists k_o such that $m(T_{k_o}) < \frac{\epsilon}{4}$. Also by [25, p. 87], there exists T^* open such that $m(T^*) < m(T_{k_o}) + \frac{\epsilon}{4} < \frac{\epsilon}{2}$

and $T^* \supset T_{k_o}$.

Define $F_o : I \to c(R^n)$ by

$$F_o(t) = \begin{cases} F(t) & t \in I - T^* \\ \{0\} & t \in T^* \end{cases} .$$

Observing that,

$$\{t : F_o(t) \cap C \neq \phi\} = \{t : F(t) \cap C \neq \phi\} \cup T^* \qquad \text{if} \quad 0 \in C$$

or

$$= \{t : F(t) \cap C \neq \phi\} \cap (I-T^*) \quad \text{if} \quad 0 \notin C$$

immediately gives that the measurability of F_o follows from that of F. Also F_o is bounded by k_o. Therefore, by lemma 2.11, there is a closed set $E' \subset I$ such that F_o restricted to E' is continuous and $m(I-E') < \frac{\epsilon}{2}$. Hence, $E = E' \cap (I-T^*)$ is the necessary closed set.

Remark: It should be noted that the above theorem and any of the later results apply equally to the usual point valued functions.

The following important selection theorem was first proved by Filippov [12]. It was later generalized by Hermes [16] and further generalized without proof by Bridgeland [6]. The following is Bridgeland's statement with a modification of Hermes' proof.

Notation: If $g : R \times R^n \to R^k$ and $A \subset R^n$ then

$$g(t,A) = \{x : x = g(t,y) \quad \text{for some} \quad y \in A\} .$$

Theorem 2.13: Let $g : R\times R^n \to R^k$ be continuous and $F : I \to c(R^n)$ be measurable. If $r : I \to R^k$ is measurable and $r(t) \in g(t,F(t))$ a.e. on I, then there exists a measurable function $v : I \to R^n$ such that $v(t) \in F(t)$ a.e. on I and $r(t) = g(t,v(t))$ a.e. on I.

Proof: For $r(t) \in g(t,F(t))$, select from those $v \in F(t)$ which satisfy $g(t,v) = r(t)$ the ones with the smallest first component. If more than one, then select from these the ones with the smallest second component, etc. The smallest values exist since g is continuous and $F(t)$ is compact, so $g(t,F(t))$ is compact. Eventually, a unique v is obtained.

Doing the above for each t, the function $v(t) = (v_1(t),\cdots, v_n(t))$ is well defined where $v_i(t)$ is the ith component of the v chosen at time t above.

The induction step and the step for $m = 1$ are identical so only the induction step is given. Therefore, assume $v_1,\cdots,v_{m-1}$ are measurable, we need only show v_m is measurable to prove the theorem.

By theorem 2.12 given $\epsilon > 0$, there exists a closed set $E \subset I$ with $m(I-E) < \epsilon$ such that $r,F,v_1,\cdots,v_{m-1}$ are continuous on E. v_m will be shown to be measurable on E by proving for any a, $\{t \in E : v_m(t) \leq a\}$ is closed, hence measurable.

If $\{t \in E : v_m(t) \leq a\}$ is not closed, then there exists $\{t_k\} \subset E$ such that $t_k \to s \in E$ and $v_m(s) > a$. E is compact and F continuous so F is bounded on E. Hence, each $v_i(t)$ is bounded on E so there exists a convergent subsequence of $\{t_k\}$, again called $\{t_k\}$,

such that for all i, $v_i(t_k) \to v_i'$ for some v_i'. Since F is continuous on E into $c(R^n)$, $v' = (v_1', \cdots, v_n') \in F(s)$.

By the continuity of $v_1(t), \cdots, v_{m-1}(t)$ on E, it follows that $v_i(s) = v_i'$, $i = 1, \cdots, m-1$. In addition, the continuity of $r(t)$ and $g(t)$ on E with the equality $g(t_k, v_1(t_k), \cdots, v_n(t_k)) = r(t_k)$ gives $g(s, v_1(s), \cdots, v_{m-1}(s), v_m', \cdots, v_n') = r(s)$. But $v_m(t_k) \leq a$ gives $v_m' \leq a < v_m(s)$ contradicting the choice of $v(s)$. Therefore v_m is measurable on E.

By a simple consequence of Luzin's Theorem [22, p. 53], if v_m is measurable on a sequence of closed sets tending to I, v_m is measurable on I which proves the induction step and the theorem.

<u>Corollary 2.14</u>: Let $F : I \to c(R^n)$ be measurable, then there exists a measurable function $f : I \to R^n$ such that $f(t) \in F(t)$ a.e.

<u>Proof</u>: In theorem 2.13, choose $g(t,x) \equiv 0$ and $r(t) \equiv 0$, then v is the desired measurable function f.

<u>Corollary 2:15</u>: Let $F : I \to c(R^n)$ and $w : I \to R^n$ be measurable, then there exists a measurable function $r : I \to R^n$ such that $r(t) \in F(t)$ a.e. and $d(w(t), r(t)) = d_1(w(t), F(t))$.

<u>Proof</u>: Define $H(t) = \{y \in F(t) : d(y, w(t)) = d_1(w(t), F(t))\}$. Since $F(t)$ is compact, $H(t)$ is nonempty. Also, by the continuity of the distance function, $H(t)$ is compact. Define

$$S(t) = \{y : d(y, w(t)) \leq d_1(w(t), F(t))\} ,$$

then $H(t) = S(t) \cap F(t)$. Since $F(t)$ is measurable, the measurability of $H(t)$ follows from that of $S(t)$, which will now be shown.

For any compact set C ,

$$\{t : S(t) \cap C \neq \phi\} = \{t : d_1(w(t),F(t)) \geq d_1(w(t),C)\} .$$

By [24, p. 93], if $d_1(w(t),F(t))$ and $d_1(w(t),D)$ are measurable, then $\{t : d_1(w(t),F(t)) \geq d_1(w(t),D)\}$ is measurable which proves $S(t)$ is measurable.

To show $d_1(w(t),F(t))$ is measurable, define

$$B_a(t) = \{y : d(y,w(t)) \leq a\} ,$$

then if $B_a(t)$ is measurable so is $F(t) \cap B_a(t)$ and

$$\{t : d_1(w(t),F(t)) \leq a\} = \{t : B_a(t) \cap F(t) \neq \phi\}$$

gives the result.

To show $B_a(t)$ is measurable, choose $D \in D^*$ with center x' and radius r' . Define $D' = \{x : d(x,x') < r' + a\}$, then

$$\{t : B_a(t) \cap D \neq \phi\} = \{t : w(t) \cap D' \neq \phi\} .$$

Thus, the measurability of $w(t)$ and lemma 2.7 proves that $B_a(t)$ is measurable. Thus by corollary 2.14 there exists a measurable function $r(t)$ such that $r(t) \in H(t)$ a.e. By construction of $H(t)$, $r(t)$ is the required function.

The above selection theorems will be used later in the existence proofs for the generalized differential equations. Next, a set of conditions are shown to be equivalent to Caratheodory type conditions for set-valued functions [6]. First, the following lemma is required.

Lemma 2.16: If $F_k : I \to c(R^n)$ are measurable, $k = 1,2,\cdots$, and $\lim_{k\to\infty} D(F_k(t),F(t)) = 0$ a.e. on I where $F : I \to c(R^n)$ then F is measurable.

Proof: Let $A = \{t : \lim_{k\to\infty} D(F_k(t), F(t)) = 0\}$.

Let $a \in R^n$ and $r \in R$ be fixed such that $\mathrm{int}(S(\{A\},r)) \in D^*$ ($\mathrm{int}(A)$ = interior of A) . For m such that $mr > 1$ define

$$T_m^k = \{t \in A : F_k(t) \cap \mathrm{int}\ (\{S(\{a\},r - \frac{1}{m})) \neq \phi\} \quad k = 1,2,\cdots$$

and

$$Z_m^n = \bigcap_{k \geq n} T_m^k \qquad n = 1,2,\cdots$$

T_m^k is measurable by lemma 2.7 and Z_m^n is measurable and hence $\bigcup_{n,m} Z_m^n$ is measurable. (See Natanson [25], p. 69, theorems 9 and 10).

We claim next that

$$T \equiv \{t \in A : F(t) \cap \mathrm{int}(S(\{a\}, r)) \neq \phi\} = \bigcup_{n,m} Z_m^n \quad \text{a.e.}$$

which gives that T is measurable.

First, it will be shown that $T \subset \bigcup_{n,m} Z_m^n$. If $t_o \in T$, then $F(t_o) \cap \mathrm{int}(S(\{a\},r)) \neq \phi$ so there exists m_o such that $m_o r > 2$ and $F(t_o) \cap \mathrm{int}(S(\{a\},r - \frac{2}{m_o})) \neq \phi$. Since $D_1(F(t_o),F_k(t_o)) \to 0$, it follows that $D_1(F(t_o) \cap S(\{a\},r - \frac{2}{m_o}), F_k(t_o)) \to 0$ as $k \to \infty$. Therefore, there exists n_o such that for $k > n_o$, $F_k(t_o) \cap \mathrm{int}(S(\{a\},r - \frac{1}{m_o}))$ $\neq \phi$. Thus $t_o \in T_{m_o}^k$ for $k \geq n_o$ so $t_o \in Z_{m_o}^{n_o}$ and $t_o \in \bigcup_{n,m} Z_m^n$.

Next, it will be shown that $\bigcup_{n,m} Z_m^n \subset T$. If $t_o \in \bigcup_{n,m} Z_m^n$, then there exists m_o and n_o such that $t_o \in Z_{m_o}^{n_o}$ so $t_o \in T_{m_o}^k$ for $k \geq n_o$. However, $D_1(F_k(t_o),F(t_o)) \to 0$ implies

$$D_1(F_k(t_o) \cap S(\{a\},r - \frac{1}{m_o}),F(t_o)) \to 0$$

so

$$F(t_o) \cap S(\{a\},r - \frac{1}{m_o}) \neq \phi$$

which certainly implies

$$F(t_o) \cap (S(\{a\},r)) \neq \phi$$

so $t_o \in T$.

But, $m(\{t \in I : F(t) \cap \mathrm{int}(S(\{a\},r)) \neq \phi\} - T) = 0$ which by lemma 2.7 implies that F is measurable.

Theorem 2.17: Let D be a non-void, open subset of $R \times R^n$ and let $F : R \times R^n \to c(R^n)$ satisfy:

(i) for every t in the projection of D on R, $F(t,\cdot)$ is continuous on $D_t = \{x \in R^n : (t,x) \in D\}$.

(ii) for every x in the projection of D on R^n and each compact interval $I \subset R$ such that $\{x\} \times I \subset D$, $F(\cdot,x)$ is measurable on I .

(iii) there exists a nonnegative function $m(t)$ which is Lebesgue integrable on R such that $\sup\{||y|| : y \in F(t,x), x \in D_t\} \leq m(t)$.

If I is a compact interval in R , and $x : I \to S \subset R^n$ is continuous such that $I \times S \subset D$, then $F(\cdot,x(\cdot))$ is measurable on I and integrably bounded.

Proof: Since I is compact and x is continuous, S can be assumed to be compact. From Dieudonne [9, section 7.6], we know x can be approximated uniformly by a sequence of step functions. Thus let $x_k(t) \to x(t)$ uniformly on I , $x_k(t)$ a step function, $k = 1,2,\cdots$ Then $x_k(t) = c_{kn}$ $t \in I_{kn}$ for intervals I_{kn} , $k = 1,2,\cdots$ and $n = 1,\cdots,m_k$, where

$$I_{kn} \cap I_{k\ell} \neq \phi \quad \text{if} \quad n \neq \ell \quad \text{and} \quad I = \bigcup_{n=1}^{m_k} I_{kn} \quad , \quad k = 1,2,\cdots$$

Given G open, define

$$M_{kn} = \{t \in I_{kn} : F(t,c_{kn}) \cap G \neq \phi\} \qquad n = 1,\cdots,m_k \quad .$$

By (ii), M_{kn} is measurable, but obviously

$$\{t \in I : F(t,x_k(t)) \cap G \neq \phi\} = \bigcup_{n=1}^{m_k} M_{kn}$$

so $F(\cdot,x_k(\cdot))$ is measurable.

By (i), $D(F(t,x_k(t)),F(t,x(t))) \to 0$ a.e. on I so by lemma 2.16, $F(\cdot,x(\cdot))$ is measurable.

Obviously, $\{||y|| : y \in F(t,x(t))\}$ is bounded by $m(t)$ which gives the bound.

Later a generalization of the integral to set-valued functions will be needed. It is defined as follows:

Definition 2.18: Given $F : [a,b] \to c(R^n)$, then for $a \leq t \leq b$,

$$\int_a^t F(s)ds \equiv \{\int_a^t f(s)ds : f : I \to R^n$$

is integrable and $f(s) \in F(s)$ a.e.$\}$

Clearly by corollary 2.14, if F is integrably bounded on $[a,b]$ and measurable, $\int_a^t F(s)\, ds$ is a nonempty subset of R^n . In fact, [1, Th. 4], it will be a compact subset of R^n under the above assumptions. The first result derived is a generalization of the equality

$$\frac{d}{dt} [\int_a^t f(s)ds] = f(t) \quad .$$

Some preliminary work is first needed.

Definition 2.19: A point of density of a measurable set $E \subset I$ is a point $t \in E$ such that

$$\lim_{h \to 0} \frac{1}{2h} [m([t-h,t+h] \cap E)] = 1 \quad .$$

Definition 2.20: Let $F : I \to c(R^n)$ be measurable. F is approximately continuous at $t \in I$ if there exists a measurable set $E \subset I$ having t as a point of density and such that the restriction of F to E is continuous.

The following lemma from [25, p. 261] will be needed.

Lemma 2.21: If E is bounded and measurable, then almost all points of E are points of density.

Proof: Let E be bounded and measurable and let $r(t)$ be the characteristic function of E. Then $r(t)$ is measurable and integrably bounded, hence integrable.

Let $e(t) = \int_a^t r(s)\, ds$, then $e'(t) = r(t)$ a.e., so $e'(t) = 1$ a.e. on E. Thus

$$\lim_{h \to 0} \frac{e(t+h) - e(t-h)}{2h} = 1 \qquad \text{a.e.}$$

But,

$$e(t+h) - e(t-h) = \int_{t-h}^{t+h} r(s)\, ds$$

$$= m(E \cap [t-h,t+h])$$

which proves the result.

Using this result, we may establish the following lemma of Hermes [15].

Lemma 2.22: If $F : I \to c(R^n)$ is measurable on $I \in c(R)$, then F is approximately continuous a.e. on I.

Proof: Given $\epsilon > 0$, by theorem 2.12, there exists $E_\epsilon \subset I$ such that E_ϵ is measurable, F is continuous on E_ϵ and $m(I-E_\epsilon) < \epsilon$. Let $D_\epsilon \subset E_\epsilon$ be the points of density of E_ϵ. Then by lemma 2.21, $m(D_\epsilon) = m(E_\epsilon) > m(I) - \epsilon$. By definition, F is approximately continuous at every $t \in D$.

Let H be the set of all points of I where F is approximately continuous. Obviously the inner measure is greater than $m(D_\epsilon)$ for all ϵ. Hence the inner measure is greater than or equal to $m(I)$. But the outer measure of H is less than or equal to $m(I)$ since $H \subset I$, and I is compact. Hence H is measurable and $m(H) = m(I)$ which proves the result.

Now we can prove the desired generalization of

$$\frac{d}{dt}\left[\int_a^t f(s)ds\right] = f(t) \quad .$$

Theorem 2.23: Let $F : [a,b] \to cc(R^n)$ be measurable and bounded by an integrable function $m : [a,b] \to R$. Let $x : [a,b] \to R^n$ be an absolutely continuous function such that $x(t) - x(s) \in \int_s^t F(s)ds$ for $s,t \in [a,b]$ then $x'(t) \in F(t)$ a.e. on $[a,b]$.

Proof: Given $\epsilon > 0$, choose $E \subset I$, E closed, such that F is continuous on E and $m(I-E) < \epsilon$. Choose K such that $||y|| < K$ for all $y \in F(t)$, $t \in E$. By lemma 2.22, F is approximately continuous a.e. on E. Choose $t^* \in E$ such that F is approximately continuous at t^* and choose $B \subset E$ such that B is measurable and t^* is a point of density of B.

Given $\delta > 0$, choose $r > 0$ such that if $t \in [t^*-r, t^*+r] \cap B$, then $D(F(t^*), F(t)) < \frac{\delta}{2}$. Since t^* is a point of density, choose $0 < h_o < r$ such that if $|h| < h_o$ then

$$\frac{1}{|h|} m([t^*, t^*+h] \cap (I-B)) < \frac{\delta}{4K} .$$

Choose $f_h : [t^*, t^*+h] \to R^n$ to be a measurable function such that $x(t^*+h) - x(t^*) = \int_{t^*}^{t^*+h} f_h(s)\, ds$ and $f_h(s) \in F(s)$ a.e.

Then given h such that $0 < |h| < h_o$,

$$\frac{1}{h}(x(t^*+h) - x(t^*)) = \frac{1}{h}\left[\int_{t^*}^{t^*+h} f_h(s)\, ds\right]$$

$$= \int_0^1 f_h(h\nabla + t^*) d\nabla \quad , \quad \text{letting } \nabla = \frac{s-t^*}{h} .$$

Clearly,

$$m\{\nabla \in [0,1] : \nabla h + t^* \in [t^*, t^*+h] \cap B\}$$

$$= \frac{1}{|h|} m\{[t^*, t^*+h] \cap B\}$$

hence, $f_h(h\nabla+t^*) \in S(F(t^*), \frac{\delta}{2})$ for all $\nabla \in [0,1]$ except on a set of measure less than $\frac{\delta}{4K}$.

Define $q : [t^*, t^*+h] \to R^n$ measurable such that

$$q(h\nabla+t^*) = f_h(h\nabla+t^*) \quad \text{if} \quad f_h(h\nabla+t^*) \in S(F(t^*), \frac{\delta}{2})$$

$$q(h\nabla+t^*) \in F(t^*) \quad \text{for all other } \nabla .$$

Hence

$$q(h\nabla+t^*) \in S(F(t^*), \frac{\delta}{2}) \quad \text{for all } \nabla .$$

So,

$$\int_0^1 q(h\nabla+t^*)d\nabla \in \text{convex hull of } S(F(t^*), \frac{\delta}{2}) ,$$

or

$$\int_0^1 q(h\nabla+t^*)d\nabla \in S(F(t^*), \frac{\delta}{2}) .$$

Noting that $f_h(h\nabla+t^*) = q(h\nabla+t^*)$ except on a set of measure less than $\frac{\delta}{4K}$ and on that set, $|f_h(h\nabla+t^*) - q(h\nabla+t^*)| < 2K$, we obtain that

$$\left|\int_0^1 q(h\nabla+t^*)d\nabla - \int_0^1 f_h(h\nabla+t^*)d\nabla\right| < \frac{\delta}{2} .$$

Thus,

$$\int_0^1 f_h(h\nabla+t^*)d\nabla \in S(f(t^*),\delta)$$

for all $0 < |h| < h_o$. Thus, $d_1(\frac{x(t^*+h) - x(t^*)}{h}, F(t^*)) < \delta$ for all $0 < |h| < h_o$. Thus, $d_1(x'(t^*),F(t^*)) < \delta$, but δ was arbitrary and $F(t^*)$ is closed, so $x'(t^*) \in F(t^*)$.

This shows that $x'(t) \in F(t)$ a.e. on E . However, the measure of E is arbitrarily near that of I , so $x'(t) \in F(t)$ a.e. on I .

The proof of the next result requires several extraneous results. For that reason, only a sketch of the proof will be presented. A complete proof is found in Aumann [1, p. 7].

<u>Theorem 2.24</u>: Let $F_k : [a,b] \to cc(R^n)$ be measurable, $k = 1,2,\cdots$, such that $||y|| \leq m(t)$ for every $y \in F_k(t)$, any k , where $m : [a,b] \to R$ is integrable, then $\limsup_{k\to\infty} F_k(t)$ is integrable and

$$\limsup_{k\to\infty} \int_a^b F_k(s)\, ds \subset \int_a^b \limsup_{k\to\infty} F_k(s)\, ds .$$

Sketch of proof:

Let $x \in \limsup_{k\to\infty} \int_a^b F_k(s)\, ds$, then there are $f_k : [a,b] \to R^n$ such that $f_k(t) \in F_k(t)$ a.e., $k = 1,2,\cdots$, and a subsequence, again called f_k , such that

$$\int_a^b f_k(s)\, ds \to x .$$

The f_k are Lebesgue integrable and uniformly bounded by an integrable function, so the f_k have a subsequence, again called f_k , converging weakly to some integrable $f : I \to R^n$ (See [10, p. 292]). Then there is a convex combination of $f_m, f_{m+1}, \cdots$, say g_m , which converges

to f in the L^1 norm (see [10, p. 422]). Hence a subsequence of the g_m, again called g_m, converges to f a.e.

Fix t, then $g_m(t)$ is a convex sum of the points $f_m(t), f_{m+1}(t), \cdots$. Recalling that all of these are in R^n and using Caratheodory's theorem [13, p. 15], there are $n+1$ of the points $f_m(t), f_{m+1}(t), \cdots$, say $e_{jm} = f_{m_j}(t)$, $j = 0, \cdots, n$ such that $g_m(t)$ is a convex sum of these points, i.e.,

$$g_m(t) = \sum_{j=0}^{m} \lambda_{jm} e_{jm} .$$

Obviously, a subsequence of the $g_m(t)$ can be formed such that $\lambda_{jm} \to \lambda_j$ and $e_{jm} \to e_j$, $j = 0, \cdots, n$, for some λ_j and e_j where

$$\sum_{j=0}^{n} \lambda_j = 1$$

and e_j is a limit of the $\{f_k(t)\}$. If $G(t)$ = all limit points of $\{f_k(t)\}$ and $G^*(t)$ = convex hull of $G(t)$, then $f(t) \in G^*(t)$. This is true for every t, so $\int_a^b f(t)\,dt \in \int_a^b G^*(t)\,dt$.

An important property of integrals of set-valued functions not established in this paper is that for any integrable set-valued function $G(t)$,

$$\int_a^b G(t)dt = \int_a^b G^*(t)dt \quad , \quad \text{if} \quad G^*(t) = \text{convex hull of } G(t) .$$

(See [6, lemma 3.3]). This shows that $\int_a^b f(t)dt \in \int_a^b G(t)dt$, but $G(t)$

is just the limit points of $\{f_k(t)\}$ so $G(t) \subset \limsup_{k\to\infty} F_k(t)$. Since

$$x = \lim_{k\to\infty} \int_a^b f_k(t)dt = \int_a^b f(t)dt \quad , \quad x \in \int_a^b \limsup_{k\to\infty} F_k(t)\, dt \quad .$$

CHAPTER III

STATEMENT OF THE PROBLEM

The necessary preliminaries have now been developed to consider the basic problem:

$$(P) \quad x'(t) \in F(t,x(t))$$

$$(P') \quad x(t_o) = x_o$$

where $F : R\times R^n \to$ subsets of R^n. Equation (P) will be called a generalized differential equation, g.d.e., and equation (P) with condition (P') will be called the generalized initial value problem, g.i.v.p.. In this paper, a solution to a g.d.e. on $E \subset R$ will be any absolutely continuous function x such that x satisfies (P) a.e. on E. A solution to the g.i.v.p. will be a solution to (P) satisfying (P'). A more restrictive definition of a solution can be taken, as in Filippov [12], but this will not be considered here.

As the problem is stated above, the g.i.v.p. need not have a solution over any interval $[t_o, t_o+h]$, since ordinary differential equations do not always have solutions. The first assumption which will be carried throughout the paper is that $F : R\times R^n \to c(R^n)$. This assumption is necessary in order that a sequence of solutions have a convergent subsequence which converges to a solution. A major, and apparently still unsolved, problem in this field is if the continuity of F with the above compactness

hypothesis is sufficient to prove the existence of a solution on some interval $[t_o, t_o+h]$. The additional assumption made is that $F : R \times R^n \to cc(R^n)$. Also certain continuity and measurability conditions will be put on F.

If F maps into one-point subsets for every (t,x), then (P) reduces to a nonlinear system of ordinary differential equations. Another class of problems, many of which are equivalent to a g.d.e., is that of differential inequalities. It will be shown next that under suitable hypotheses, problem (P) is equivalent to the paratingent problem stated in the first chapter. This equivalence was first observed and proved by Wazewski [31]. First a lemma must be established.

Lemma 3.1: If $S \in cc(R^n)$, x is absolutely continuous on $[a,b]$, and $x'(t) \in S$ a.e. on $[a,b]$, then $\dfrac{x(b) - x(a)}{b - a} \in S$.

Proof: Choose $\epsilon > 0$. Since S is compact, it can be covered by a finite number of open spheres, $S_1, \cdots, S_q$, with radii of less than $\epsilon/2$. Define

$$M_1 = \overline{S}_1$$

and

$$M_i = \overline{S}_i \cap \left(\bigcup_{j=1}^{i-1} (R^n - S_j) \right) \qquad i = 2, \cdots, q .$$

Then each M_i is closed and the interiors of the M_i are mutually disjoint. Define

$$P_i = S \cap M_i \qquad i = 1,\cdots,q$$

and

$$Q_1 = P_1$$

and

$$Q_i = P_i \cap (\bigcup_{j=1}^{i-1} R^n - P_j) \quad .$$

Then $S = \bigcup_{i=1}^{q} Q_i$, $Q_i \cap Q_j = \phi$ if $i \neq j$, and the diameter of Q_i is less than ϵ .

Using the Q_i construct

$$T_i = \{t \in [a,b] : x'(t) \in Q_i\}$$

$$= \{t \in [a,b] : x'(t) \in P_i\} \cap (\bigcap_{j=1}^{i-1} \{t \in [a,b] : x'(t) \notin P_j\}) \ .$$

Since P_j is closed for each j , T_i is measurable for $i = 1,\cdots,q$. Also, the T_i are mutually disjoint and

$$\sum_{i=1}^{q} m(T_i) = b-a \ , \quad \text{or} \quad \sum_{i=1}^{q} \left(\frac{m(T_i)}{b-a}\right) = 1 \quad .$$

Choose $r_i \in P_i \subset S$, then by convexity of S ,

$$\sum_{i=1}^{q} \left(\frac{m(T_i)}{b-a}\right) r_i \in S \quad .$$

However,

$$\left| \int_{T_i} x'(s)ds - r_i m(T_i) \right| < \epsilon \, m(T_i) .$$

So,

$$\left| \frac{1}{b-a} \int_a^b x'(s)ds - \sum_{i=1}^{q} \left(\frac{m(T_i)}{b-a}\right) r_i \right| < \epsilon ,$$

which proves the result recalling that S is closed and ϵ arbitrary, and that

$$\int_a^b x'(s)ds = x(b) - x(a) .$$

Theorem 3.2: Let W be an open subset of $R \times R^n$. If $F : W \to cc(R^n)$ is u.s.c. and if $E(t,x)$ is the set of all straight lines passing through (t,x) with slope in $F(t,x)$, then the following are equivalent:

(1) (a) $x(t)$ is continuous on $J \subset R$.

(b) The paratingent of $x(t)$ at t is contained in $E(t,x(t))$ on J.

(2) (a) $x(t)$ is absolutely continuous on J.

(b) $x'(t) \in F(t,x(t))$ a.e.

Proof: Assume (1) holds on compact subsets I of J, then by (1a) and hypothesis $F(t,x(t))$ is u.s.c. in t. For every $t \in I$, define $D_t = (t-\delta_t, t+\delta_t)$ where δ_t is such that $S(F(t),1) \supset F(s)$ for all $s \in D_t$. By compactness, a finite number of the D_t 's cover I, say $D_{t_1}, \cdots, D_{t_n}$. Then $C = \bigcup_{i=1}^{n} \overline{D}_{t_i}$ is a compact set such that $F(s) \subset C$

for all $s \in I$. Hence, $\left|\frac{x(t) - x(s)}{t - s}\right|$ remains uniformly bounded on I , so $x(t)$ is Lipschitz continuous, hence $x(t)$ is absolutely continuous on I . J is a countable union of compact sets from which (2a) follows. Thus $x'(t)$ exists a.e. from which (2b) follows.

Assume (2) holds, then (1a) follows immediately. Choose $t_o \in J$ and $\epsilon > 0$, then $S(F(t_o,x(t_o)),\epsilon) \in cc(R^n)$. By u.s.c., there is a δ such that for every $t \in (t_o-\delta,t_o+\delta)$

$$F(t,x(t)) \subset S(F(t_o,x(t_o)),\epsilon)$$

so

$$x'(t) \in S(F(t_o,x(t_o)),\epsilon) \quad \text{a.e. on} \quad (t_o-\delta,t_o+\delta) \quad .$$

Thus by lemma 3.1, $\dfrac{x(t) - x(t_o)}{t - t_o} \in S(F(t_o,x(t_o)),\epsilon)$ if $t \in (t_o-\delta,t_o+\delta)$. Thus the paratingent of x at t_o is contained in the set of straight lines through $(t_o,x(t_o))$ with slope in $S(F(t_o,x(t_o)),\epsilon)$. ϵ is arbitrary which proves (2b) .

A second, and more important, equivalent problem occurs in control theory. The basic problem here is given a set of measurable controls $u(t)$, find a solution to $x'(t) = f(t,x(t),u(t))$ for some $u(t)$ in this set. The proof of the following theorem is motivated by Hermes and LaSalle [17].

<u>Theorem 3.3</u>: Let $f : R\times R^n\times R^m \to R^n$ be continuous and $C : R\times R^n \to c(R^m)$ be measurable. If F is defined by

$$F(t,x) = \{f(t,x,u) : u \in C(t,x)\}$$

then the following two problems are equivalent:

(1) (a) $x(t) = \int_a^t f(s,x(s),u(s))ds$

(b) $u(t)$ is measurable on J

(c) $u(t) \in C(t,x(t))$ a.e. on J

(2) (a) $x'(t) \in F(t,x(t))$ a.e. on J

(b) $x(t)$ is absolutely continuous.

Proof: Assume x satisfies (1), then (2b) is immediately satisfied and $x'(t) = f(t,x(t),u(t))$ obtained by differentiating (1a) gives (2a).

Assume x satisfies (2). Define

$$g(t,u) = f(t,x(t),u)$$

and

$$F(t) = C(t,x(t)) \quad .$$

Applying theorem 2.13 with $r(t) = x'(t)$ gives that there exists a measurable function $u(t) \in F(t) = C(t,x(t))$ a.e. such that $x'(t) = f(t,x(t),u(t))$ a.e.. Hence (1) holds.

CHAPTER IV

BASIC THEORY

The basic theory of g.d.e.'s will now be derived. Many approaches have been used. Hermes [16] and Bridgeland [5] consider the case where $F : R^{n+1} \to cc(R^n)$ is continuous and remains uniformly bounded. Castaing [7] and Plis [27] extended the theory to functions $F : R^{n+1} \to cc(R^n)$ which are integrably bounded and only measurable in t. They proved existence through fixed point theorems. Bridgeland [6] and Kikuchi [20,21] later did this more general problem using a constructive approach. Daures [8] has further generalized the problem to functions mapping appropriate topological spaces into closed subsets of a Banach space. Finally, Roxin [29] developed the theory considering g.d.e.'s as a generalized dynamical system. The treatment that follows in this paper is closest to that of Kikuchi.

Two basic sets of hypotheses will be imposed on the g.d.e.'s. These are closely related and given as HC and HO below:

(HC) $F : [a,b] \times R^n \to cc(R^n)$ such that

(i) For every $t \in [a,b]$, $F(t,\cdot)$ is continuous on R^n.

(ii) For every $x \in R^n$, $F(\cdot,x)$ is measurable on $[a,b]$.

(iii) There exists a Lebesgue integrable function $m : [a,b] \to R$ such that $\sup\{||y|| : y \in F(t,x), x \in R^n\} \leq m(t)$.

(HO) $F : [a,\infty) \to cc(R^n)$ such that

(i) For every $t \in [a,\infty)$, $F(t,\cdot)$ is continuous on R^n .

(ii) For every $x \in R^n$ and compact interval $I \subset R$, $F(\cdot,x)$ is measurable on I .

(iii) There exists a Lebesgue integrable function $m : [a,\infty) \to R$ such that $\sup\{||y|| : y \in F(t,x) , x \in R^n\} \leq m(t)$.

It will be clear in the following that F could be defined in less restrictive domains than the infinite slabs above. In this case, a local theory would go through with only slight modifications in the proofs, but the global theory desired could only be obtained with stringent additional hypotheses to insure that solutions would not leave the domain of definition.

By corollary 2.10, if F is continuous then (ii) in both sets of hypotheses is satisfied, so all the theory developed below applies to an integrably bounded continuous function. The basic existence theorems will now be stated and proved.

Theorem 4.1: Let F satisfy (HC) , then for any $t_o \in [a,b]$ and $x_o \in R^n$ there exists a solution to

$$\text{(P)} \quad x'(t) \in F(t,x(t)) \quad \text{a.e.}$$

$$\text{(P')} \quad x(t_o) = x_o \quad ,$$

existing on $[a,b]$.

Proof: For simplicity, t_o will be assumed to be a. Similar arguments go through if this is not the case.

For each positive integer k, define $h_k = \frac{b-a}{k}$. Define $f(t) \equiv 0$ for $t \in [a - h_k, a]$. Construct $x_k : [a,b] \to R^n$ as follows:

(1) For $a \leq t \leq a + h_k$, define $x_k(t) \equiv x_o$. Then select a measurable function f such that $f(t) \in F(t,x_k(t))$ a.e. on $[a,a+h_k]$. This is possible by corollary 2.14 and theorem 2.17.

(2) Assume x_k and f have been defined on $[a,a+(n-1)h_k]$. For $t \in [a+(n-1)h_k, a+nh_k]$ define

$$x_k(t) = x_k(a + (n-1)h_k) + \int_{a+(n-2)h_k}^{t-h_k} f(s)\, ds .$$

Then define f on $[a + (n-1)h_k , a + nh_k]$ by choosing a measurable selection $f(t) \in F(t,x_k(t))$ a.e.

(3) Continue the above process until $n = k$. This is possible by (iii).

Therefore given $t \in [a,b]$, for some n between 0 and k, $t \in [a + (n-1)h_k , a + nh_k]$. Thus

$$x_k(t) = x_k(a + (n-1)h_k) + \int_{a+(n-2)h_k}^{t-h_k} f(s)\, ds$$

$$x_k(t) = x_k(a + (n-2)h_k) + \int_{a+(n-3)h_k}^{a+(n-2)h_k} f(s)\, ds + \int_{a+(n-2)h_k}^{t-h_k} f(s)\, ds$$

etc., or

(4.1)
$$x_k(t) = x_o + \int_{a-h_k}^{t-h_k} f(s)\, ds \quad .$$

Obviously x_k is continuous. In addition, the following estimates hold.

$$|x_k(t) - x_o| = |\int_{a-h_k}^{t-h_k} f(s)\, ds|$$
$$\leq \int_{a}^{t-h_k} |f(s)|\, ds$$

or

(4.2)
$$|x_k(t) - x_o| \leq \int_{a}^{t-h_k} m(s)\, ds \leq \int_{a}^{b} m(s)\, ds \quad .$$

So $\{x_k\}$ form a bounded family on $[a,b]$.

Claim 1: $\{x_k\}$ form an equicontinuous family on $[a,b]$.

Proof of claim:

Given $s \neq a$ then if $t \in [a,b]$,

$$|x_k(t) - x_k(s)| = |\int_{a-h_k}^{t-h_k} f(u)du - \int_{a-h_k}^{s-h_k} f(u)\, du|$$
$$\leq |\int_{s-h_k}^{t-h_k} |f(u)|du|$$
$$\leq |\int_{s-h_k}^{t-h_k} m(u)\, du| \quad .$$

Choose K' such that $s-a > \frac{1}{K'}$. Then given $\epsilon > 0$, choose δ such that $s-a - \frac{1}{K'} > \delta > 0$ and

$$\int_{s-2\delta}^{\min(s+\delta,b)} m(s)\, ds < \epsilon .$$

Choose $K^* > K'$ such that if $k > K^*$, then $h_k = \frac{b-a}{k} < \delta$. Let t be such that $|s-t| < \delta$.

If $t > s$ and $t \in [a,b]$, then

$$\int_{s-h_k}^{t-h_k} m(u)\, du \leq \int_{s-\delta}^{\min(t,s+\delta-h_k)} m(u)\, du$$

$$\leq \int_{s-\delta}^{\min(b,s+\delta)} m(u)\, du < \epsilon .$$

If $s > t$, then

$$\int_{t-h_k}^{s-h_k} m(u)\, du \leq \int_{s-\delta-h_k}^{s} m(u)\, du$$

$$\leq \int_{s-2\delta}^{s} m(u)\, du < \epsilon .$$

Hence $|x_k(t) - x_k(s)| < \epsilon$ if $|t-s| < \delta$ and $k > K^*$. For all $k_i \leq K^*$, $i = 1,\cdots,m$, there exists δ_i such that if $|t-s| < \delta_i$ then $|x_{k_i}(t) - x_{k_i}(s)| < \epsilon$.

Choosing $\delta' = \min(\delta_1,\cdots,\delta_m,\delta)$ equicontinuity follows for all $s \neq a$. If $s = a$, equicontinuity follows from (4.2) which proves the result.

Therefore by Ascoli's theorem, there is a convergent subsequence, again called x_k , such that $x_k(t) \to x(t)$ uniformly on $[a,b]$. Obviously,

$x(a) = x_o$ since this holds for each $x_k(a)$.

Claim 2: $x(t) \in x(s) + \int_s^t F(u,x(u))\,du$.

Proof of the claim:

From (4.1) it follows that

$$x_k(t) \in x_k(s) + \int_{s-h_k}^{t-h_k} F(u,x_k(u))\,du$$

for each k . So

$$x(t) \in x(s) + \limsup_{k\to\infty} \int_{s-h_k}^{t-h_k} F(u,x_k(u))\,du$$

$$\subset x(s) + \int_s^t \limsup_{k\to\infty} F(u,x_k(u))\,du$$

$$+ \limsup_{k\to\infty} \int_t^{t+h_k} F(u,x_k(u))\,du$$

$$- \limsup_{k\to\infty} \int_s^{s-h_k} F(u,x_k(u))\,du$$

(by theorem 2.24).

The claim follows from (i) and the fact that $x_k(t) \to x(t)$ on $[a,b]$.

Claim 3: $x(t)$ is absolutely continuous.

Proof of the claim:

$x(t) - x(s) \in \int_s^t F(u,x(u))\,du$ by claim 2, hence

$$x(t) - x(s) = \int_s^t f(u)\,du \quad \text{where} \quad f(u) \in F(u,x(u)) \quad \text{a.e.}$$

So

$$|x(t) - x(s)| \leq \left|\int_s^t f(u)\,du\right| \leq \left|\int_s^t m(u)\,du\right|$$

for any $t,s \in [a,b]$.

Given $\epsilon > 0$, choose $\delta > 0$ such that for any n and $\{t_i,t_i^*\}_{i=1}^n$ with $\sum_{i=1}^n |t_i^*-t_i| < \delta$, it follows that $\sum_{i=1}^n \left|\int_{t_i}^{t_i^*} m(s)\,ds\right| < \epsilon$. But,

$$\sum_{i=1}^n \left|\int_{t_i}^{t_i^*} x(s)\,ds\right| \leq \sum_{i=1}^n \left|\int_{t_i}^{t_i^*} m(s)\,ds\right| < \epsilon$$

which proves the claim.

Therefore, by theorem 2.23, $x'(t) \in F(t,x(t))$ a.e. on $[a,b]$ which proves the theorem.

<u>Lemma 4.2</u>: Let F satisfy (H0) , and let x be an absolutely continuous function such that $x'(t) \in F(t,x(t))$ a.e. on $[a,c)$, then x can be extended continuously to $[a,c]$.

<u>Proof</u>: Since $|x(t) - x(s)| = \left|\int_s^t x'(u)\,du\right|$, it follows that

$$|x(t) - x(s)| \leq \left|\int_s^t m(u)\,du\right| \quad . \tag{4.3}$$

If $t,s \to c^-$, then $\int_s^t m(u)\,du \to 0$ so $|x(t) - x(s)| \to 0$ and hence

$\lim_{t \to c^-} x(t)$ exists.

Lemma 4.3: Let F satisfy (HO), then any solution x to (P)(P'), defined on $[a,a+\epsilon]$, $\epsilon > 0$, can be extended as a solution to (P) (P') to $[a,\infty)$.

Proof: Let $\{(I,x_I)\}$ be the set of all extensions of x such that $[a,a+\epsilon] \subset I \subset [a,\infty)$ I an interval, $x_I(t) = x(t)$ if $t \in [a,a+\epsilon]$ and x_I is a solution to (P)(P') on I. Let $\leq$ be a partial order of $\{(I,x_I)\}$ where $(I,x_I) \leq (J,x_J)$ if $I \subset J$ and $x_I(t) = x_J(t)$ for $t \in I$.

Let $\{(I_\alpha,x_{I_\alpha})\}$ be a totally ordered subset. Then define $I = \cup_\alpha I_\alpha$ and $x_I(t) = \{x_{I_\alpha}(t) : t \in I_\alpha\}$. x_I is well-defined since the x_{I_α} agree on any common interval of existence. Then (I,x_I) is an upper bound for the totally ordered subset since $[a,a+\epsilon] \subset I$ and $x_I(t) = x(t)$ if $t \in [a,a+\epsilon]$. So by Zorn's lemma, the set of all extensions of $x(t)$ has a maximal element. Hence for some interval I, x_I is a non-extendable extension of x.

If $I = [a,b)$, $b < \infty$, then by lemma 4.2 x_I can be extended to $[a,b]$ continuously and hence will be a solution to (P)(P') on $[a,b]$ contradicting the maximality of $[a,b)$.

If $I = [a,b]$, then for $d > b$ F satisfies hypotheses HC on $[b,d]$ with condition (P_b') being $x(b) = x_I(b)$. Hence by theorem 4.1, there is a solution x_b to $(P)(P_b')$ on $[b,d]$. Defining

$$x(t) = \begin{cases} x_I(t) & t \in I = [a,b] \\ x_b(t) & t \in (b,d] \end{cases}$$

clearly exhibits an extension to $x_I(t)$ to $[a,d]$ contradicting the maximality of $[a,b]$.

Therefore $I = [a,\infty)$ which proves the lemma.

Theorem 4.4: Let F satisfy (HO) , then for any $t_o \in [a,\infty)$ and $x_o \in R^n$ there exists a solution to (P)(P') on $[a,\infty)$.

Proof: Choose $b > t_o$, then by theorem 4.1 there is a solution x to (P)(P') existing on $[a,b]$. By lemma 4.3, this solution can be extended to $[a,\infty)$.

The next result gives the desirable conclusion that a sequence of solutions to (P) has a subsequence which converges to a solution of (P) . A somewhat more general lemma will be useful later.

Lemma 4.5: Let F satisfy (HC) and let $x_i : [s_i,t_i] \to R^n$ and $y_i : [s_i,t_i] \to R^n$, $[s_i,t_i] \subset [a,b]$, be such that x_i is absolutely continuous, $x_i'(t) \in F(t,y_i(t))$ a.e., and $x_i(t) \to x(t)$ and $y_i(t) \to x(t)$ for some function $x : [s^*,t^*] \to R^n$ where $s_i \to s^*$ and $t_i \to t^*$, then $x'(t) \in F(t,x(t))$ a.e. on $[s^*,t^*]$.

Proof: The proof is similar to the last part of the proof of theorem 4.1. We have that $x_i(t) = x_i(s) + \int_s^t x_i'(u)\,du$ where $x_i'(u) \in F(u,y(u))$ a.e. so using an argument as in claim 2 of theorem 4.1, it follows that

$$x(t) \in x(s) + \int_s^t F(u,x(u))\, du \quad .$$

By claim 3 of theorem 4.1, x is absolutely continuous, thus theorem 2.23 proves the lemma.

Corollary 4.6: Let F satisfy (HC) and let $\{x_i\}$ be a sequence of solutions of (P) on $[a,b]$ such that $x_i(s_i) \in A$ for some compact set A and some sequence $\{s_i\}$ of points in $[a,b]$, then there is a subsequence of $\{x_i\}$ which converges uniformly to x, a solution to (P) on $[a,b]$.

Proof: The proof is similar to the first part of theorem 4.1. Choose M such that $||y|| < M$ for all $y \in A$, then each x_i is bounded by $M + \int_a^b m(s)\, ds$ on $[a,b]$.

The relation $|x_i(t) - x_i(s)| \leq |\int_s^t m(u)\, du|$ holds so an argument similar to that proving claim 1 of theorem 4.1 gives the equicontinuity of $\{x_i\}$. Therefore, by Ascoli's theorem, there is a convergent subsequence such that $x_i(t) \to x(t)$ uniformly on $[a,b]$. The corollary follows by the previous lemma.

Corollary 4.7: Let F satisfy (HO) and let $\{x_i\}$ be a sequence of solutions to (P) on $[a,\infty)$ such that $x_i(s_i) \in A$ for some compact set A and some bounded sequence $\{s_i\}$ in $[a,\infty)$, then there is a subsequence which converges to x, a solution of (P), uniformly on compact subsets.

Proof: Choose an integer m such that $\{s_i\}$ is contained in $[a,m]$.

Define $I_n = [a,n]$, $n = m, m+1, \cdots$.

Choose a subsequence $x_{m,i}$ which converges to a solution x on I_m . This exists by corollary 4.6. Assuming $x_{n,i}$ is a subsequence of $\{x_i\}$ converging to a solution x on I_n , choose a subsequence of $\{x_{n,i}\}$, say $\{x_{n+1,i}\}$, converging to a solution x on I_{n+1} . Again, this is possible for each n by corollary 4.6.

The diagonal sequence $x_{i,i}$ is the desired subsequence.

In general, unique solutions to g.d.e.'s are not to be expected. In discussing the behavior of solutions, it will be useful to consider the set of points reachable from a given point or set by solutions.

Definition 4.8: $A(t,t_o,B)$ is the attainable set for F if:

$$A(t_1,t_o,B) = \{x(t_1) : x(t_o) \in B \quad \text{and} \quad x'(t) \in F(t,x(t)) \quad \text{a.e.}$$

$$\text{on} \quad [\min(t_o,t_1) \ , \ \max(t_o,t_1)]\} \ .$$

For short, $A(t,t_o,\{x_o\})$ will be denoted by $A(t,t_o,x_o)$.

Some basic properties of attainable sets will be required in later work. The basic hypotheses under which these are proven are given below:

(H) F satisfies (HC) and $I = [a,b]$ or
F satisfies (HO) and $I = [a,\infty)$.

Lemma 4.9: If F and I are as in (H) , and if $t_o,t_1,s \in I$ such that $t_o \leq s \leq t_1$ or $t_1 \leq s \leq t_o$ then $A(t_1,t_o,B) = A(t_1,s,A(s,t_o,B))$.

Proof: We will assume $t_o \leq s \leq t_1$. The proof for the other case is similar.

Let $y \in A(t_1,t_o,B)$, and let x be the solution such that $y = x(t_1)$ and $x(t_o) \in B$. Then $x(s) \in A(s,t_o,B)$ so $y \in A(t_1,s, A(s,t_o,B))$.

Let $y \in A(t_1,s,A(s,t_o,B))$, then let x_1 be a solution such that $y = x_1(t_1)$ and $x_1(s) \in A(s,t_o,B)$. Then let x_2 be a solution such that $x_2(t_o) \in B$ and $x_2(s) = x_1(s)$. Then

$$x(t) = \begin{cases} x_1(t) & t_1 \geq t \geq s \\ x_2(t) & s > t \geq t_o \end{cases}$$

is a solution to (P) such that $y = x(t_1)$ and $x(t_o) \in B$ so $y \in A(t_1,t_o,B)$.

Theorem 4.10: Let F and I be as in (H), and let B be a compact subset of R^n, then $A(t,t_o,B)$ is compact for all $t_o,t \in I$.

Proof: Let $\{y_k\}$ be a sequence in $A(t_1,t_o,B)$. Then there exist solutions x_k such that $x_k(t_o) \in B$ and $x_k(t_1) = y_k$. By corollary 4.6 or corollary 4.7, a subsequence of $\{x_k\}$ converges to a solution x. By compactness, $x(t_o) \in B$, so $x(t_1) \in A(t_1,t_o,B)$. A subsequence of $\{y_k\}$ converges to $x(t_1)$ which proves the theorem.

Theorem 4.11: Let F and I be as in (H), and let $t_o \in I$ and B compact be fixed, then $A(\cdot,t_o,B) \to c(R^n)$ is u.s.c.

Proof: Let $s \in I$ and $\epsilon > 0$ be given. Choose $\delta > 0$ such that $\left|\int_{I\cap[s-\delta,s+\delta]} m(u)du\right| < \epsilon$. If x is a solution to (P) , $t \in I$, and $|t-s| < \delta$, then by (4.3) $|x(t) - x(s)| < \epsilon$. Thus by lemma 4.9, $A(t,t_o,B) \subset S(A(s,a,B),\epsilon)$ if $|t-s| < \delta$ proving the result.

Notation: If F and I are as in (H) and if $w : I \to R^n$ is measurable, then given a continuous function x , the relation:

$$d_1(w\ (t),F(t,x(t))) = d(w(t),u(t)) \tag{4.4}$$

with $u(t) \in F(t,x(t))$ defines uniquely a measurable function u . To see this, recall from corollary 2.15 that there exists a measurable function satisfying (4.4). That u is well-defined, note that if the closed ball of radius $d_1(w(t),F(t,x(t)))$ with center $w(t)$ intersects $F(t,x(t))$ at two points y_1 and y_2 then by convexity

$$sy_1 + (1-s)y_2 \in F(t,x(t)) \quad , \quad 0 \leq s \leq 1 ,$$

which contradicts the definition of the ball. Denote the function corresponding to x by u_x .

Lemma 4.12: Let F and I be as in (H) , and let $w : I \to R^n$ be a measurable function, then for any continuous functions x and y mapping I into R^n and $\epsilon > 0$ and small enough, there exists $\delta > 0$ such that if $d(x(t),y(t)) < \delta$ then $d(u_x(t),u_y(t)) < \epsilon$. (Note that δ and ϵ depend on t and x , but not y).

<u>Proof</u>: Let $S = \{x : d(x,u_x(t)) \geq \frac{\epsilon}{2}\}$, then define $S_1 = S \cap F(t,x(t))$. Hence S_1 is compact. If $F(t,x(t))$ contains more than one point, then for ϵ small enough, S_1 is not empty.

<u>Case 1</u>: $F(t,x(t))$ contains more than one point.

By construction, $d(w(t),S_1) > d(w(t),u_x(t))$ so there exists δ' such that $d_1(w(t),S_1) > d(w(t),u_x(t)) + 2\delta'$ since S_1 is compact. Define $\delta_1 = \min(\frac{\epsilon}{2},\delta')$. Choose δ such that $d(x(t),y(t)) < \delta$ implies $D(F(t,x(t)),F(t,y(t))) < \delta_1$. Therefore, there exists $a \in F(t,y(t))$ and $b \in F(t,x(t))$ such that $d(a,u_x(t)) < \delta_1$ and $d(b,u_y(t)) < \delta_1$. Hence,

$$d(w(t),u_y(t)) = d_1(w(t),F(t,y(t)))$$

$$\leq d(w(t),a)$$

$$d(w(t),u_y(t)) \leq d(w(t),u_x(t)) + d(u_x(t),a)$$

$$< d(w(t),u_x(t)) + \delta_1 \quad .$$

Also,

$$d(w(t),b) \leq d(w(t),u_y(t)) + d(u_y(t),b)$$

$$< d(w(t),u_y(t)) + \delta_1$$

$$< d(w(t),u_x(t)) + 2\,\delta_1 \quad .$$

Hence $b \notin S_1$ by definition of δ'. Thus $d(u_x(t),b) < \frac{\epsilon}{2}$. Also $d(b,u_y(t)) < \delta_1 < \frac{\epsilon}{2}$. Thus $d(u_x(t),u_y(t)) < \epsilon$.

Case 2: $F(t,x(t))$ is a point.

In this case $u_x(t) = f(t,x(t))$. The result follows since $F(t,\cdot)$ is continuous.

Theorem 4.13: Let F and I be as in (H) , $t_o \in I$, and B be a compact, connected subset of R^n , then $A(t,t_o,B)$ is connected.

Proof: Assume $A(t_1,t_o,B)$ is not connected. By theorem 4.10, it is compact so there exist compact, non-empty subsets S_1 and S_2 such that $A(t_1,t_o,B) = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$.

Define $B_i = A(t_o,t_1,S_i) \cap B$, $i = 1,2$. Then by theorem 4.10, B_i is compact and obviously $B = B_1 \cup B_2$. B is connected, so $B_1 \cap B_2 \neq \phi$. Let $x_o \in B_1 \cap B_2$. Define $S_i' = A(t_1,t_o,x_o) \cap S_i$, $i = 1,2.$ Then S_1' and S_2' are non-empty, compact subsets of R^n such that $S_1' \cap S_2' = \phi$. Hence, $\mathrm{dist}(S_1',S_2') = d > 0$. Define $H = \{x:d_1(x,S_1') = \frac{d}{2}\}$. Then, $H \cap S_i' = \phi$, $i = 1,2$.

Choose $x_i \in S_i'$, $i = 1,2$. Then choose $x_i : I \to R^n$ such that $x_i(t_o) = x_o$, $x_i(t_1) = x_i$, and $x_i'(t) \in F(t,x_i(t))$ a.e..

Given any integer k , let $h_k = \dfrac{t_1-t_o}{k}$ and define $t_{n,k} = t_o + (n-1)\, h_k$. Construct $x_{k,i}(t,t^*)$ for each $t^* \in [t_o,t_1]$, $i = 1,2$ and $k = 1,2,\cdots$, as follows:

(1) For $t \in [t_o,t^*]$, define $x_{k,i}(t,t^*) = x_i(t)$.

(2) Find n such that $t_{n,k} \leq t^* < t_{n+1,k}$, then define

$$g_{n-1}(t) = \begin{cases} x_{k,i}(t,t^*) & t \in [t_o,t^*] \\ x_{k,i}(t^*,t^*) & t \in (t^*,t_{n+1,k}] \end{cases}$$

$g_{n-1}(t)$ is absolutely continuous so $u_n(t) \equiv u_{g_{n-1}}(t)$ exists satisfying (4.4) with $w(t) = x_i'(t)$ and $x(t) = g_{n-1}(t)$. For $t \in [t^*,t_{n+1,k}]$ define:

$$x_{k,i}(t,t^*) = x_{k,i}(t^*,t^*) + \int_{t^*}^{t} u_n(s)\,ds \quad .$$

Also define:

$$g_n(t) = \begin{cases} x_{k,i}(t,t^*) & t \in [t_o,t_{n+1,k}] \\ x_{k,i}(t_{n+1,k},t^*) & t \in (t_{n+1,k}\, , \, t_{n+2,k}] \quad . \end{cases}$$

(3) Define $\{u_m\}$, $x_{m,i}(\cdot,t^*)$, and $\{g_m\}$ on $[t_o,t_{m+1,k}]$, $m = n,n+1,\cdots,k$, by defining $u_m(t) \equiv u_{g_{m-1}}(t)$ satisfying (4.4) with $w(t) = x_i'(t)$ and $x(t) = g_{m-1}(t)$. Then define:

$$x_{k,i}(t,t^*) = x_{k,i}(t^*,t^*) + \int_{t^*}^{t} u_m(s)\,ds \quad \text{on} \quad [t_{m,k},t_{m+1,k}] \quad .$$

Finally define:

$$g_m(t) = \begin{cases} x_{k,i}(t,t^*) & t \in [t_o,t_{m+1,k}] \\ x_{k,i}(t_{m+1,k},t^*) & t \in (t_{m+1,k}\, , \, t_{m+2,k}] \quad . \end{cases}$$

The above procedure defines $x_{k,i}(\cdot,t^*)$ for $i = 1,2$ and $k = 1,2,\cdots$, and $x(\cdot,t^*)$ is absolutely continuous by construction.

Claim 1: $x_{k,i}(t,\cdot)$ is continuous.

Proof of claim:

Choose t^* and find n such that $t_{n-1,k} \le t^* < t_{n,k}$.

<u>Case 1</u>: $t < t^*$.

If $t' < t^*$ also, then $x_{k,i}(t,t^*) = x_{k,i}(t,t') = x_i(t)$. Thus $|x_{k,i}(t,t^*) - x_{k,i}(t,t')| = 0$ if $d(t,t') < d(t^*,t)$.

<u>Case 2</u>: $t = t^*$.

If $t' \geq t^*$, $|x_{k,i}(t,t^*) - x_{k,i}(t,t')| = 0$. If $t' < t^*$, then

$$|x_{k,i}(t,t^*) - x_{k,i}(t,t')| \le |x_i(t^*) - x_i(t')| + \left|\int_{t'}^{t^*} u_n(s)\, ds\right|$$

$$\le |x_i(t^*) - x_i(t')| + \int_{t'}^{t^*} m(s)\, ds$$

which shows continuity.

<u>Case 3</u>: $t > t^*$.

If $t_{n-1,k} = t^*$, go directly to the induction hypothesis. Otherwise, let

$$d_1 = \min(d(t^*,t_{n-1,k}), d(t^*,t_{n,k}))$$

and choose $s \in (t^*, t_{n,k}]$. If $d(t',t^*) < d$, then

$$|x_{k,i}(s,t^*) - x_{k,i}(s,t')| \leq |x_i(t^*) - x_i(t')| + \left|\int_{t'}^{t^*} u_n(r)\, dr\right|$$

$$\leq |x_i(t^*) - x_i(t')| + \left|\int_{t'}^{t^*} m(r)\, dr\right|$$

where $u_n(r)$ is the function associated with $x_{k,i}(s,t^*)$ or $x_{k,i}(s,t')$, whichever is appropriate. Thus $x_{k,i}(s,\cdot)$ is continuous for $s \leq t_{n,k}$.

Induction hypothesis: Assume $x_{k,i}(s,\cdot)$ is continuous at t^* for $s \in [t_o, t_{m,k}]$, then it is continuous for $s \in [t_o, t_{m+1,k}]$ if $m \leq k-1$.

Proof of induction hypothesis: For convenience assume $t^* \neq t_{n-1,k}$. If $t^* = t_{n-1,k}$, the argument below works with slight modification.

Again choose t' such that $d(t',t^*) < d$ and $s \in (t_{m,k}, t_{m+1,k}]$. Let g_m^* and g_m' be the associated g functions in the above construction defined on $[t_o, t_{m+1,k}]$. By the continuity of $x_{k,i}(s,\cdot)$ at t^* on $[t_o, t_{m,k}]$, given $\epsilon > 0$ there exists $\delta, d_1 > \delta > 0$, such that $d(t',t^*) < \delta$ implies

$$|g_m^*(t) - g_m'(t)| < \epsilon \qquad \text{for all} \qquad t \in [t_o, t_{n+1,k}] \ .$$

Hence by lemma 4.12, $|u_{g_m'}(t) - u_{g_m^*}(t)| \to 0$ a.e. as $d(t',t^*) \to 0$. Noting that $|u_{g_m'}(t) - u_{g_m^*}(t)| \leq 2m(t)$ a.e. and using the Lebesgue dominated convergence theorem [25, p. 161], it follows that $\left|\int_{t'}^{s} (u_{g_m'}(r) - u_{g_m^*}(r))dr\right| \to 0$ as $d(t',t^*) \to 0$. Hence,

$$|x_{k,i}(t,t^*) - x_{k,i}(t,t')| \leq |x_i(t^*) - x_i(t')| + \left|\int_{t^*}^{t'} u_{g_n^*}(r)\, dr\right|$$

$$+ \left|\int_{t'}^{s} (u_{g_n'}(r) - u_{g_n^*}(r))\, dr\right|$$

which proves continuity on $(t_{m,k}, t_{m+1,k}]$. Thus the induction hypothesis is true from which the claim follows.

Define

$$y_k(s) = \begin{cases} x_{k,1}(t_1, t_o + (t_o - t_1)s) & -1 \leq s \leq 0 \\ x_{k,2}(t_1, t_o + (t_1 - t_o)s) & 0 < s \leq 1 \end{cases} .$$

Then $y_k : [-1,1] \to R^n$ is continuous for each k and $y_k(-1) = x_1$ and $y_k(1) = x_2$. Thus for some s_k , $y_k(s_k) \in H$.

Define

$$z_k(t) = \begin{cases} x_{k,1}(t, t_o + (t_o - t_1)s_k) & s_k \leq 0 \\ x_{k,2}(t, t_o + (t_1 - t_o)s_k) & s_k > 0 \end{cases} .$$

Also define

$$w_k(t) = \begin{cases} x_1(t) & t \in [t_o, t_o + (t_o - t_1)s_k] & s_k \leq 0 \\ x_2(t) & t \in [t_o, t_o + (t_o - t_1)s_k] & s_k > 0 \\ x_1(t_o + (t_o - t_1)s_k) & t \in [t_o + (t_o - t_1)s_k, t_{m,k}] & s_k \leq 0 \\ x_2(t_o + (t_1 - t_o)s_k) & t \in [t_o + (t_1 - t_o)s_k, t_{m,k}] & s_k > 0 \\ x_k(t_{n+1,k}, t_o + (t_1 - t_o)|s_k|) & t \in [t_{n+1,k}, t_{n+2,k}] & m \leq n \leq k-1 \end{cases}$$

where $t_o + (t_1-t_o)s_k \in [t_{m-1,k}, t_{m,k})$. Then $z_k'(t) \in F(t,w_k(t))$ a.e. on $[t_o,t_1]$. Hence,

$$|z_k(t)| \leq |z_k(t_o)| + \left|\int_{t_o}^{t_1} m(s)\, ds\right|$$

which shows that the $\{z_k\}$ are uniformly bounded since

$$\begin{aligned} |z_k(t_o)| &= |x_{k,i}(t_o, t_o + (t_1-t_o)|s_k|)| \\ &= |x_i(t_o + (t_1-t_o)|s_k|)| \\ &\leq |x_i(t_1)| \\ &\leq |x_i(t_o)| + \left|\int_{t_o}^{t_1} m(s)\, ds\right| \\ &\leq |x_o| + \left|\int_{t_o}^{t_1} m(s)\, ds\right| . \end{aligned}$$

Also, $|z_k(t) - z_k(s)| \leq \left|\int_s^t m(u)\, du\right|$ from which equicontinuity follows. Hence a subsequence, again called z_k , converges to a function $z : [t_o,t_1] \to R^n$. Since $s_k \in [-1,1]$ for all k , a subsequence of these, called s_k again, converges to some point $s \in [-1,1]$. From these two observations, it follows that $w_k(t) \to w(t)$ for some $w : I \to R^n$. If $t \leq t_o + (t_1-t_o)\ |s_k|$ then w_k and z_k tend to the function x_i where $i = 1$ or 2 . If $t > t_o + (t_1-t_o)|s_k|$ then

$$|w_k(t) - z_k(t)| = |x_k(t_{n+1,k}, t_o + (t_o-t_1)|s_k|) - x_k(t, t_o + (t_o-t_1)|s_k|)|$$

where $t \in [t_{n+1,k}, t_{n+2,k}]$, but $d(t_{n+1,k}, t_{n+2,k}) \to 0$ so $d(w_k(t), z_k(t)) \to 0$.

Hence $z(t) = w(t)$. Thus by lemma 4.5, $z'(t) \in F(t,z(t))$ a.e.. Obviously, $z(t_o) = x_o$ and $z(t_1) \in H$. Hence $z(t_1) \in S_i'$ for $i = 1$ or 2 which contradicts $H \cap S_i' = \phi$ for $i = 1$ and 2 .

CHAPTER V

FURTHER INVESTIGATION

The previous section shows that the solutions to generalized differential equations behave much like those of ordinary differential equations with nonunique solutions. The rest of this paper will deal with a somewhat more specific problem that has been considered in the case of ordinary differential equations. Some work has also been done in the case of generalized differential equations. It is concerned with the behavior of solutions in some subset V of $R \times R^n$. Of interest is giving subsets V in which either one solution remains in V for all time or some solution is in V at all times. Three approaches to the problem are considered.

The first approach is similar to that used in the two dimensional case by Bebernes and Wilhelmsen [2,3]. In this case, the dimension of R^n must be greater than or equal to 2. Let $x \equiv (x_1, \cdots, x_n) \in R^n$ and let $y \equiv (x_2, \cdots, x_n)$, then V will be the region bounded by the surfaces $x_1 = w(t,y)$ and $x_1 = z(t,y)$, that is

$$V = \{(x_1, \cdots, x_n) : w(t,y) \leq x_1 \leq z(t,y)\} .$$

The following hypotheses will be imposed.

(HS) 1. $w(t,y) < z(t,y)$ for every $(t,y) \in R^n$.

2. w and z are continuous functions.

3. For every $t_o \in [a,\infty)$ and $y_o \in R^{n-1}$ there exists $\epsilon > 0$ and a solution x to (P)(P') with $x_o = (z(t_o,y_o),y_o)$ $(x_o = (w(t_o,y_o),y_o))$ such that $x_1(t) > z(t,x_2(t),\cdots,x_n(t))$ $(x_1(t) < w(t,x_2(t),\cdots,x_n(t))$ for $t \in [t_o , t_o+\epsilon]$, where $x(t) = (x_1(t),\cdots,x_n(t))$.

With the above hypotheses, the surfaces corresponding to w and z can be viewed as generalizations of the upper-solutions and lower-solutions used in considering two point boundary value problems [2]. A convenient notation is:

$$\text{(N)} \quad W(s) = \{(x_1,\cdots,x_n) : x_1 = w(s,x_2,\cdots,x_n)\}$$

$$Z(s) = \{(x_1,\cdots,x_n) : x_1 = z(s,x_2,\cdots,x_n)\}$$

$$C(s) = \{(x_1,\cdots,x_n) : w(s,x_2,\cdots,x_n) \leq x_1 \leq z(s,x_2,\cdots,x_n)\}.$$

The following generalization of Theorem 1 [3] can now be stated and proved.

Theorem 5.1: Let S_1 be a compact, connected set in $C(t_o)$ which intersects $W(t_o)$ and $Z(t_o)$. Let F satisfy (HO) and let w and z satisfy (HS), then $A(t,t_o,S_1) \cap C(t)$ contains a compact, connected component intersecting both $W(t)$ and $Z(t)$ for all t.

Proof: Let $x : [a,\infty) \to R^n$ be a solution to (P) such that $x(t_o) \in S_1$, then

$$|x(t)| \leq K + \int_{t_o}^{t} m(s)\, ds \tag{5.1}$$

where $S_1 \subset S((0,\cdots,0),K)$.

Consider the interval $[t_o,t_o+n]$ and define

$$K_n = K + \int_{t_o}^{t_o+n} m(s)\, ds$$

and $t_n = t_o + n$. If the conclusion of the theorem is true for all $t \in [t_o,t_o+n]$, $n = 1,2,\cdots$, then the conclusion of the theorem is true for all t which proves the theorem.

Choose any n and let $I = [t_o,t_n]$. Define:

$T = \{t^* \in I$: if $t \in [t_o,t^*]$, $A(t,t_o,S_1) \cap C(t)$ contains a compact, connected component intersecting $W(t)$ and $Z(t)\}$.

By the hypotheses, $t_o \in T$ so T is nonvoid. Let $s = \sup\{t : t \in T\}$, then it will be first shown that T is closed so $s \in T$.

Let $s_i \in T$, $i = 1,2,\cdots$, and assume $s_i \to s$. Let $C_i \subset C(s_i)$ be a compact, connected component of $A(s_i,t_o,S_1)$ intersecting $W(s_i)$ and $Z(s_i)$. We define the limit set L of a sequence of sets $\{D_i\}$ to be

$L = \{x$: for any neighborhood N of x, $N \cap D_i \neq \phi$ for infinitely many i's$\}$.

Let L be the limit set of the C_i 's.

Claim 1: $L \cap W(s) \neq \phi$.

Proof of claim:

$C_i \cap W(s_i) \neq \phi$ for each s_i so choose $x_i \in C_i \cap W(s_i)$. By (5.1), $|x_i| \leq K_n$ so there is a convergent subsequence, again called x_i,

converging to some a . Obviously $a \in L$. Also $x_1^i = w(s_i, x_2^i, \cdots, x_n^i)$ where $x_i = (x_1^i, x_2^i, \cdots, x_n^i)$ so by continuity of w $a \in W(s)$. Thus $a \in L \cap W(s)$.

Consider the subsequence C_i corresponding to the subsequence $x_i \to a$. Let L_1 be the limit set of this subsequence. By arguments analogous to claim 1, $L_1 \cap W(s) \neq \phi$ and $L_1 \cap Z(s) \neq \phi$.

Claim 2: L_1 is compact.

Proof of claim:

Let $x_j \in L_1$, $j = 1,2,\cdots$, then for each j there is a sequence $x_{jk} \in C_{i_k}$, $i_k > k$, such that $x_{jk} \to x_j$. Then $\{x_{jj}\}$ is a bounded sequence so it has a convergent subsequence, again called $\{x_{jj}\}$, $x_{jj} \to x$ and $x_{jj} \in C_{i_j}$ where $i_j \to \infty$. Hence $x \in L_1$ and a subsequence of $\{x_j\}$ converges to it proving the claim.

Claim 3: L_1 is connected.

Proof of claim:

Assume not, then $L_1 = M \cup N$ where M and N are nonvoid, disjoint, compact subsets of R^n . Hence by Urysohn's lemma [32, lemma 15.6] there exists a continuous function $f : R^n \to [0,1]$ such that

$$f(x) = 0 \quad \text{if} \quad x \in M$$

$$f(x) = 1 \quad \text{if} \quad x \in N \ .$$

Consider $A = \{x : f(x) = \frac{1}{2}\}$. Obviously A is closed. Define

$A_1 = A \cap \{x : |x| \leq K_n\}$, then A_1 is compact.

Without loss of generality, assume $a \in M$. Choose $d \in N$. Let C_{i_k} be a subsequence of $\{C_i\}$ such that there is some $y_{i_k} \in C_{i_k}$ such that $y_{i_k} \to d$. Then $x_{i_k} \to a$ since $\{x_i\}$ does. If i_k is large enough, $x_{i_k} \in \{x : f(x) < \frac{1}{4}\}$ and $y_{i_k} \in \{x : f(x) > \frac{3}{4}\}$. Recalling that C_{i_k} is connected and observing that

$$C_{i_k} = (\{x : f(x) \geq \frac{1}{2}\} \cap C_{i_k}) \cup (\{x : f(x) \leq \frac{1}{2}\} \cap C_{i_k}) \quad ,$$

it is clear that for some $z_{i_k} \in C_{i_k}$ that $f(z_{i_k}) = \frac{1}{2}$, otherwise C_{i_k} is the union of two nonempty, disjoint, compact subsets. Since $|z_{i_k}| \leq K_n$, $z_{i_k} \in A_1$ and by the compactness of A_1 there is a subsequence converging to some $z \in A_1$. Since $z_{i_k} \in C_{i_k}$, $z \in L_1$ and by continuity $f(z) = \frac{1}{2}$ which is a contradiction.

Claim 4: $L_1 \subset A(s,t_o,S_1)$.

Proof of claim:

If $x \in L_1$, then there exists some subsequence $x_i \in C_i$ such that $x_i \to x$. But $C_i \subset A(s_i,t_o,S_1)$ so there is a sequence of solutions $\{y_i\}$ such that $y_i : I \to R^n$, $y_i'(t) \in F(t,y_i(t))$ a.e. on I , $y_i(t_o) \in S_1$ and $y_i(s_i) = x_i$. By corollary 4.6, a subsequence converges to a solution $y : I \to R^n$. Since $x_i \to x$, $y(s) = x$ and $y(t_o) \in S_1$ since S_1 is compact. Thus $x \in A(s,t_o,S_1)$.

The above claims show that L_1 is the required connected component so $s \in T$ and hence T is closed.

If $s = t_n$, then the conclusion of the theorem holds on I and hence the theorem is proved. Hence assume $s < t_n$. By theorem 4.10 and theorem 4.13, $A(t,s,L_1)$ is compact and connected. By construction of L_1 , choose $x_w \in L_1 \cap W(s)$ and $x_z \in L_1 \cap Z(s)$.

Hypothesis (HS) implies that there exists $\epsilon > 0$ and solutions x and y to (P) such that

1. $x(s) = x_w$ and $y(s) = x_z$.

2. $x_1(t) \leq w(t,x_2(t),\cdots,x_n(t))$ for all $t \in [s,s+\epsilon]$ where $x(t) = (x_1(t),\cdots,x_n(t))$.

3. $y_1(t) \geq z(t,y_2(t),\cdots,y_n(t))$ for all $t \in [s,s+\epsilon]$ where $y(t) = (y_1(t),\cdots,y_n(t))$.

Thus for each $t \in [s,s+\epsilon]$ define

$$W_1(t) = W(t) \cap \{x : |x| \leq K_n\}$$

and

$$Z_1(t) = Z(t) \cap \{x : |x| \leq K_n\} .$$

Then $W_1(t)$ and $Z_1(t)$ are compact, connected sets. $Z_1(t) \cap A(t,s,L_1) \neq \phi$ and $W_1(t) \cap A(t,s,L_1) \neq \phi$ otherwise $Z(t)$ or $W(t)$ would disconnect $A(t,s,L_1)$.

By [32, Th. 26.7], $D \equiv W_1(t) \cup Z_1(t) \cup A(t,s,L_1)$ is a compact, connected set. Let $A(t,s,L_1) \cap C(t)$ be decomposed into connected components D_α , then these are compact [32, Th. 26.12].

If for some α , $D_\alpha \cap W_1(t) = \phi$ and $D_\alpha \cap Z_1(t) = \phi$, then D_α is a compact, connected component of $A(t,s,L_1)$ which is not equal to $A(t,s,L_1)$. This contradicts the fact that $A(t,s,L_1)$ is connected.

Define $Z_2(t) = \cup \{D_\alpha : D_\alpha \cap Z_1(t) \neq \phi\}$ and $W_2(t) = \cup \{D_\alpha : D_\alpha \cap W_1(t) \neq \phi\}$. Then $A(t,s,L_1) \cap C(t) = W_2(t) \cup Z_2(t)$. If $x \in W_2(t) \cap Z_2(t)$ then $x \in D_\alpha$ such that $D_\alpha \cap Z_1(t) \neq \phi$ and $D_\alpha \cap W_1(t) \neq \phi$ since all the D_α are disjoint. Thus D_α is a component of $A(t,s,L_1)$ such that it intersects $W(t)$ and $Z(t)$.

If $x \notin Z_2(t) \cap W_2(t)$ but $x \in Z_2(t) \cap \overline{W}_2(t)$ then there exists $\{D_{\alpha_i}\}$ and $\{x_i\}$ such that $x_i \in D_{\alpha_i}$ such that $x_i \to x$ where $D_{\alpha_i} \subset W_2(t)$ and $x \in D_\beta \subset Z_2(t)$. Let E be the limit set of the D_{α_i} , then by an argument similar to that of claim 3 E is connected. By continuity, $E \cap W_1(t) \neq \phi$. Also $x \in E$. Since $A(t,s,L_1) \cap C(t)$ is closed, $E \subset A(t,s,L_1) \cap C(t)$. Therefore $E \subset W_2(t)$ and $x \in Z_2(t) \cap W_2(t)$ which contradicts the assumption that $x \notin Z_2(t) \cap W_2(t)$. Similarly, if $x \in \overline{Z}_2(t) \cap W_2(t)$, it follows that $x \in W_2(t) \cap Z_2(t)$.

Hence if there is no compact, connected set in $A(t,s,L_1) \cap C(t)$ intersecting $W_1(t)$ and $Z_1(t)$, it follows that $Z_2(t)$ and $W_2(t)$ are separated. For $D_\alpha \in Z_2(t)$, $D_\alpha \cup Z_1(t)$ is connected thus

$Z_3(t) = Z_1(t) \cup Z_2(t)$ is connected. Likewise, $W_3(t) = W_1(t) \cup W_2(t)$ is connected. Also $W_3(t)$ and $Z_3(t)$ are separated.

Consider $\{x : x_1 \geq z(t,x_2,\cdots,x_n)\}$. Let Z_α be the connected components of this set. Again if $Z_\alpha \cap Z_1(t) = \phi$ then Z_α is a disconnected subset of $A(t,s,L_1)$. Hence $Z_3(t) \cup \{x : x_1 \geq z(t,x_2,\cdots,x_n)\}$ is a compact connected subset. Likewise $W_3(t) \cup \{x : x_1 \leq w(t,x_2,\cdots,x_n)\}$ is compact and connected. The above sets are separated which implies $A(t,s,L_1)$ is separated. This is a contradiction so there is a compact, connected subset in $A(t,s,L_1) \cap C(t)$ for every $t \in [s,s+\epsilon]$. This contradicts the maximality of s so $s = t_o + n$ and the theorem is proved.

The second result is a generalization of a topological principle of Wazewski [14, p. 278]. This result insures that at least one solution starting in some set of initial values $S \subset V$ remains in V for all time if whenever any solution leaves V it leaves for some finite time. Conditions are imposed on the set points V_e where some solution leaves V, S, and $V_e \cap S$. This result was generalized to contingent equations whose vector fields are upper semicontinuous by Bebernes and Schuur [4]. Here it will be shown that a slightly different generalization remains valid for generalized differential equations satisfying (HO). First some definitions are required to formulate this principle. For this work, V will be assumed to be an open subset of $[a,\infty) \times R^n$.

<u>Definition 5.2:</u> (a) A point $(t_1,x_1) \in \partial V$ is a consequent of a point $(t_o,x_o) \in V$ if there exists $x : I \to R^n$ such that $x'(t) \in F(t,x(t))$ a.e. on I, $(t,x(t)) \in V$ if $t \in [t_o,t_1)$, $x(t_o) = x_o$, and $x(t_1) = x_1$.

(b) A point $(t_o,x_o) \in \partial V$ is a consequent of itself if there exists a solution to (P) (P') .

(c) The consequent mapping $C : \overline{V} \to$ subsets of ∂V is defined by $C(t,x) = \{(t_1,x_1) : (t_1,x_1)$ is a consequent of $(t,x)\}$.

Definition 5.3: A point $(t_1,x_1) \in \partial V$ is a strict consequent point if for every solution $x : I \to R^n$ of (P) with $x(t_1) = x_1$, there exists a sequence $t_n \to t_1 + 0$ such that $(t_n,x(t_n)) \notin \overline{V}$.

Definition 5.4: A solution x to (P) such that $(t_o,x(t_o)) \in V$ leaves V if there exists some $t_1 > t_o$ such that $(t_1,x(t_1)) \notin V$.

It will now be shown that C maps $\overline{V}$ into $c(R^n)$, that $C(t,x)$ is connected, and that C is u.s.c. if all solutions leave V and if all consequent points are strict. Under the above conditions and further assumptions on V , a contradiction will be obtained proving the existence of a solution which does not leave V .

Lemma 5.5: Let $(t_o,x_o) \in \overline{V}$ such that all solutions through (t_o,x_o) leave V and all consequent points are strict, then if F satisfies (HO) , $C(t_o,x_o)$ is compact.

Proof: Let $(t_n,x_n) \in C(t_o,x_o)$, $n = 1,2,\cdots$, then there exists a solution $x_n : [a,\infty) \to R^n$ to (P) such that $x_n(t_o) = x_o$, $x_n(t_n) = x_n$, and $(t,x_n(t)) \in V$ for $t \in [t_o,t_n)$. By corollary 4.7, a subsequence of these solutions converge to a solution x and $x(t_o) = x_o$.

By hypothesis x leaves V, say at time t^*, so $(t^*,x(t^*)) \notin \overline{V}$. Then for n large enough, $x_n(t)$ leaves V before time t^* since $x_n(t^*) \to x(t^*) \notin \overline{V}$. Thus $t_n \in [t_o,t^*]$, hence there is a convergent subsequence, again called t_n, such that $t_n \to t'$ and by continuity $(t_n,x_n(t_n)) \to (t',x(t'))$.

Since $(t_n,x_n(t_n)) \in \partial V$ and for any set its boundary is closed, it follows that $(t',x(t')) \in \partial V$. If there exists s such that $t_o \leq s < t'$ and $(s,x(s)) \notin V$ then because consequent points are strict for some s_o such that $s < s_o < t'$ we have $(s_o,x(s_o)) \notin \overline{V}$ and thus for n large enough $(s_o,x_n(s_o)) \notin \overline{V}$. However also for n large enough, $t_n > s_o$ which contradicts that $(t_n,x_n(t_n)) \in C(t_o,x_o)$. Thus $(t,x(t)) \in V$ for $t \in [t_o,t')$ so $(t',x(t')) \in C(t_o,x_o)$.

In lemma 5.7 we will use implicitly the concept of weak invariance of Roxin [28] and Yorke [33] although it will not be necessary to develop the general theory. The proof of the following lemma [33, Th. 3.6] is motivated by a proof of Roxin [28, lemma 7.1].

Lemma 5.6: If R and S are nonempty closed subsets relative to V with $V = R \cup S$ such that through every point of $R(S)$ there exists at least one solution through the point which remains in $R(S)$ and if F satisfies (HO), then given $(t^*,x^*) \in R \cap S$, there exists $\epsilon > 0$ and a solution x such that $x(t^*) = x^*$ and $(t,x(t)) \in R \cap S$ if $t \in [t^*,t^*+\epsilon]$.

Proof: By theorem 4.10 and theorem 4.13, $A(t,t^*,x^*)$ is compact and connected for all $t \geq t^*$. Since $(t^*,x^*) \in V$ which is open, there exists $\delta > 0$ such that $S((t^*,x^*),\delta) \subset V$. Choose ϵ such that $0 < \epsilon < \frac{\delta}{2}$ and

$$\int_{t^*}^{t^*+\epsilon} m(s)\, ds < \frac{\delta}{2} \text{ , then if } t \in [t^*,t^*+\epsilon] \text{ , } \{t\} \times A(t,t^*,x^*) \subset V \text{ .}$$

Any solution starting at (t_1,x_1) , $x_1 \in A(t_1,t^*,x^*)$, can be extended to time $t^* + \epsilon$ in V . If $(t_1,x_1) \in R \cap S$, at least one solution remains in R and at least one in S . Hence

$$R_1 \equiv [\{t\} \times A(t,t_1,x_1)] \cap R \neq \phi \quad \text{and} \quad S_1 \equiv [\{t\} \times A(t,t_1,x_1)] \cap S \neq \phi$$

for $t \in [t_1,t^*+\epsilon]$. Also, R_1 and S_1 are closed relative to V since they are both the intersection of two closed sets. Finally, $\{t\} \times A(t,t_1,x_1) = R_1 \cup S_1$ and since $\{t\} \times A(t,t_1,x_1)$ is connected this implies $R_1 \cap S_1 \neq \phi$ or $[\{t\} \times A(t,t_1,x_1)] \cap (R \cap S) \neq \phi$ for each $t \in [t_1,t^*+\epsilon]$.

Using the fact just established, we will now construct a sequence of solutions of (P) such that they converge to a solution of (P) lying in $R \cap S$.

1. Choose $(t^*+\epsilon,x_{oo}) \in [\{t^*+\epsilon\} \times A(t^*+\epsilon,t^*,x^*)] \cap (R \cap S)$ and choose $x_o : [t^*,t^*+\epsilon] \to R^n$ as a solution to (P) satisfying $x_o(t^*) = x^*$ and $x_o(t^*+\epsilon) = x_{oo}$.

2. Choose $(t^* + \frac{\epsilon}{2} , x_{10}) \in [\{t^* + \frac{\epsilon}{2}\} \times A(t^* + \frac{\epsilon}{2} , t^*,x^*)] \cap (R \cap S)$ and $(t^*+\epsilon,x_{11}) \in [\{t^*+\epsilon\} \times A(t^*+\epsilon,t^* + \frac{\epsilon}{2} , x_{10})] \cap (R \cap S)$. Choose $z_1 : [t^*,t^* + \frac{\epsilon}{2}] \to R^n$ and $z_2 : [t^* + \frac{\epsilon}{2} , t^*+\epsilon] \to R^n$ as solutions to (P) satisfying

$$z_1(t^*) = x^* \qquad z_1(t^* + \tfrac{\epsilon}{2}) = x_{10}$$

$$z_2(t^*) = x_{10} \qquad z_2(t^* + \epsilon) = x_{11}$$

and define

$$x_1(t) = \begin{cases} z_1(t) & t \in [t^*, t^* + \frac{\epsilon}{2}) \\ z_2(t) & t \in [t^* + \frac{\epsilon}{2}, t^* + \epsilon] . \end{cases}$$

3. Continue as in 2 choosing $x_n : [t^*, t^*+\epsilon] \to R^n$ such that $x_n(t^*) = x^*$,

$$(t^* + \frac{m\epsilon}{2^n}, x_n(t^* + \frac{m\epsilon}{2^n})) \in [\{t^* + \frac{m\epsilon}{2^n}\} \times$$

$$A(t^* + \frac{m\epsilon}{2^n}, t^* + \frac{(m-1)\epsilon}{2^n}, x_n(t^* + \frac{(m-1)\epsilon}{2^n})] \cap (R \cap S) ,$$

$m = 1,2,\cdots,2^n$, and x_n is a solution to (P) .

By corollary 4.7, a subsequence of these solutions converges to a solution $x : [t^*, t^*+\epsilon] \to R^n$. For any point $t_{m,i} = t^* + \frac{m\epsilon}{2^i}$, $0 \leq m \leq 2^i$ and i and m positive integers, if n is large enough $(t_{m,i}, x_n(t_{m,i})) \in R \cap S$. But $R \cap S$ is closed and $(t_{m,i}, x(t_{m,i})) \in V$ so $(t_{m,i}, x(t_{m,i})) \in R \cap S$. The points $t_{m,i}$ are dense in $[t^*, t^*+\epsilon]$ so by continuity, $(t, x(t)) \in R \cap S$ for all $t \in [t^*, t^*+\epsilon]$.

Lemma 5.7: If $(t^*, x^*) \in \overline{V}$ and if all solutions through (t^*, x^*) leave V and all consequent points are strict, then $C(t^*, x^*)$ is connected.

<u>Proof</u>: If $(t^*,x^*) \in \partial V$, then $C(t^*,x^*) = \{(t^*,x^*)\}$ and the lemma is true. If $(t^*,x^*) \in V$, assume the lemma is false. Then by lemma 5.5, there are disjoint, non-empty, compact sets C_1 and C_2 such that $C(t^*,x^*) = C_1 \cup C_2$.

Given a solution z to (P) with $z(s_1) = y$, define $F(z,s_1,y) = \{(s,z(s)) :$ for all $t \in [\min(s,s_1), \max(s,s_1)]$, $(t,z(t)) \in V\}$.

Define:

$R = \cup\{F(z,t,x) : (t,x) \in V$ and z is some solution to (P) with $z(t) = x$ and $\mathrm{dist}(F(z,t,x),C_1) \leq \mathrm{dist}(F(z,t,x),C_2)\}$.

$S = \cup\{F(z,t,x) : (t,x) \in V$ and z is some solution to (P) with $z(t) = x$ and $\mathrm{dist}(F(z,t,x),C_2) \leq \mathrm{dist}(F(z,t,x),C_1)\}$.

By the definition of consequent points and the compactness of C_1 and C_2, if

$$\mathrm{dist}(F(z,t^*,x^*),C_1) = 0 \quad \text{then} \quad \mathrm{dist}(F(z,t^*,x^*),C_2) \neq 0 .$$

Also, since C_1 and C_2 are nonempty, R and S are nonempty.

Claim 1: R is closed relative to V.

Proof of claim:

Let $(t_n,x_n) \in R$, $n = 1,2,\cdots$, such that $(t_n,x_n) \to (t_o,x_o) \in V$. Then there exists a solution z_n to (P) with $z_n(t_n) = x_n$ and

$\text{dist}(F(z_n,t_n,x_n),C_1) \leq \text{dist}(F(z_n,t_n,x_n),C_2)$. By corollary 4.7, a subsequence again called z_n , converges to a solution z . Define

$$s_+(z,s_1) = \sup\{s : (t,z(t)) \in V \quad \text{for} \quad t \in [s_1,s)\} \ .$$

First we will show that

$$s^* \equiv \liminf_{n\to\infty} s_+(z_n,t_n) \geq s_+(z,t_o) \ .$$

If n is large, since $(t_n,z_n(t_n)) \to (t_o,x_o) \in V$, $(t,z_n(t)) \in V$ if $t \leq t_o$. Hence $s^* \geq t_o$. However $z_n(s^*) \to z(s^*)$ but there is a subsequence of $z_n(s^*)$ such that $\lim_{n\to\infty}(s^*,z_n(s^*)) \in R^n - V$ which is closed, hence $(s^*,z(s^*)) \in R^n - V$. Thus, $s^* \geq s_+(z,t_o)$. Second, we will show that $s_+(z,t_o) \geq \limsup_{n\to\infty} s_+(z_n,t_n)$. If not, there exists s^* such that $s_+(z,t_o) < s^* < \limsup_{n\to\infty} s_+(z_n,t_n)$ such that $(s^*,z(s^*)) \notin \overline{V}$. However $(s^*,z_n(s^*)) \to (s^*,z(s^*))$ which implies for large n , $(s^*,z_n(s^*)) \notin \overline{V}$ which implies $s_+(z_n,t_n) \leq s^*$ for large n which is a contradiction. Hence, $s_+(z,t_o) = \lim_{n\to\infty} s_+(z_n,t_n)$. From this, the continuity of $F(z,\cdot,\cdot)$ follows and thus the claim.

A similar argument proves that S is closed relative to V .

Through every point of V there exists at least one trajectory which must belong to R or S or both, so $V = R \cup S$. We also have that $(t^*,x^*) \in R \cap S$ since there are two solutions y_1 and y_2 through the point with $\text{dist}(F(y_1,t^*,x^*),C_1) = 0$ and $\text{dist}(F(y_2,t^*,x^*),C_2) = 0$.

Claim 2: If $(s,y) \in R$, there is at least one solution to (P) through (s,y) which remains in R until it leaves V.

Proof of claim:

There exists a solution z such that $z(s) = y$ and $\text{dist}(F(z,s,y),C_1) \leq \text{dist}(F(z,s,y),C_2)$. If $(s_1,y_1) = (s_1,z(s_1))$ before the solution leaves V then $\text{dist}(F(z,s,y),C_i) = \text{dist}(F(z,s_1,y_1),C_i)$, $i = 1,2$. So $(s_1,y_1) \in R$.

Again, by a similar argument, if $(s,y) \in S$, there is at least one solution to (P) through (s,y) which remains in S until it leaves V.

Claim 3: There exists a solution through (t^*,x^*) which remains in $R \cap S$ until it leaves V.

Proof of claim:

By lemma 5.6, for some $\epsilon > 0$ there exists a solution z to (P) through (t^*,x^*) remaining in $R \cap S$ for $t \in [t^*,t^*+\epsilon]$. Assume no solution remains in $R \cap S$ until it leaves V.

By an argument similar to that of lemma 4.3, there exists some right maximal interval $I \supset [t^*,t^*+\epsilon]$ such that some extension z^* of z remains in $R \cap S$ on I. Since z^* leaves V, I is a finite interval with right endpoint t'. Since z^* is continuous and $R \cap S$ is closed, $I = [t^*,t']$. However, by lemma 5.6, a solution z_1 remains in $R \cap S$ on $[t',t'+\epsilon]$ with $z_1(t') = z^*(t')$. Hence

$$y(t) = \begin{cases} z^*(t) & t^* \leq t \leq t' \\ z_1(t) & t' < t \leq t'+\epsilon \end{cases}$$

remains in $R \cap S$ on $[t^*, t'+\epsilon]$ contradicting the maximality of I. Hence, some solution starting at (t^*, x^*) remains in $R \cap S$ until it leaves V, proving the claim.

Consider this solution z such that $z(t^*) = x^*$ which remains in $R \cap S$, we obtain $\mathrm{dist}(F(z,t^*,x^*),C_1) = \mathrm{dist}(F(z,t^*,x^*),C_2) = 0$ since it remains in both sets and $C(t^*,x^*) = C_1 \cup C_2$. This contradicts the observation before claim 1, thus proving the lemma.

<u>Lemma 5.8</u>: If all solutions from $A \subset V$ leave V and if all consequent points are strict, then $C : R \times R^n \to c(R^n)$ is u.s.c..

<u>Proof</u>: If C is not u.s.c. at $(t_o, x_o) \in A$ then there exists $\epsilon > 0$ and $\{(t_n, x_n)\} \subset A$ such that $(t_n, x_n) \to (t_o, x_o)$ and $C(t_n, x_n)$ is not contained in $S(C(t_o, x_o), \epsilon)$.

Hence, there exists $(t_n', x_n') \in C(t_n, x_n)$ such that $d_1((t_n', x_n'), C(t_o, x_o)) > \epsilon$. So there exists solutions z_n such that $z_n(t_n') = x_n'$ and $z_n(t_n) = x_n$ and $(t, z_n(t)) \in V$ if $t \in [t_n, t_n']$.

By corollary 4.7, a subsequence of the z_n, again called z_n, converges to a solution z of (P). By assumption z leaves V, say at time t^* for the first time. Choose $\epsilon > 0$. Then for n large enough, $t_n' \leq t^* + \epsilon$. Hence, there is a convergent subsequence of the t_n', again called t_n'. By an argument as in claim 1 of lemma 5.7, $t_n' \to t^*$ and thus

$z_n(t_n') \to z(t^*)$ which is a contradiction since $(t^*, z(t^*)) \in C(t_o, x_o)$. This proves the lemma.

A general lemma on upper semicontinuous functions will be needed.

Lemma 5.9: If $C : A \to c(B)$, $B \subset R^n$, is u.s.c. on A , if $C(x)$ is compact and connected for all $x \in A$, and if $D \subset A$ is compact and connected, then $C(D)$ is compact and connected.

Proof: First, we will show that $C(D)$ is compact. For every $x_o \in D$, choose ϵ_{x_o} such that $C(x) \subset S(C(x_o),1)$ if $d(x,x_o) < \epsilon_{x_o}$. The spheres $\{y : d(x,y) < \epsilon_x\}$ form an open cover of D , hence there is a finite subcover, corresponding to $x_1,\cdots,x_n$. Thus $C(D) \subset \bigcup_{i=1}^{n} S(C(x_i),1)$ so $C(D)$ is bounded. If $\{x_n\} \subset C(D)$ then it is bounded so it has a subsequence, again $\{x_n\}$, such that $x_n \to x_o \in R^n$. Also there exists $\{y_n\} \subset D$ such that $x_n \in C(y_n)$. There is a subsequence, again called y_n , such that $y_n \to y_o \in D$. For every $\epsilon > 0$, there exists N such that for $n > N$ $C(y_n) \subset S(C(y_o),\frac{\epsilon}{2})$. If N is also large enough that $d(x_o,x_n) < \frac{\epsilon}{2}$, then $x_o \in S(C(y_o),\epsilon)$. Since $C(y_o)$ is closed and ϵ arbitrary, $x_o \in C(y_o)$ so $x_o \in C(D)$.

Second, we will show that $C(D)$ is connected. Assume $C(D)$ is not connected, then $C(D) = C_1 \cup C_2$ where C_1 and C_2 are disjoint, nonvoid, compact subsets. Then for every $x \in D$, $C(x) \subset C_i$, $i = 1$ or 2 . [32, 26.6].

Define D_1 and D_2 by

$$D_i = \{x : C(x) \subset C_i\} \qquad i = 1,2 \ .$$

Since each C_i is nonempty, each D_i is nonempty. Also $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \phi$.

Since D is connected, the result follows if each D_i is closed. Therefore, let $\{x_n\} \subset D_i$, $x_n \to x_o$. Given any $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x_n, x_o) < \delta$ then $C(x_n) \subset S(C(x_o), \epsilon)$. This implies $C_i \subset S(C(x_o), \epsilon)$ for every ϵ . Since C_i and $C(x_o)$ are both closed, $C(x_o) \cap C_i \neq \phi$. Hence $C(x_o) \subset C_i$ and $x_o \in D_i$ so D_i is closed.

To formulate the Wazewski topological principle, a new concept is necessary. In point-set topology, $B \subset A$ is said to be a retract of A if there is a continuous function $g : A \to B$ such that $g(b) = b$ for each $b \in B$. [32, p. 224]. We will need to generalize this to set-valued functions.

<u>Definition 5.10</u>: If A and B are subsets of R^{n+1} with $B \subset A$ then B is a set-valued retract of A if there exists an u.s.c. mapping $G : A \to c(R^{n+1})$ such that $G(x) \subset B$ for $x \in A$, $G(x)$ is connected for $x \in A$, and $x \in G(x)$ for $x \in B$.

If B is a retract of A , then B is a set-valued retract of A . The converse is not true. The unit circle is not a retract of the unit disk [32, Th. 34.5], however, the following theorem shows that it is a set-valued retract of the unit disk.

Theorem 5.11: If $A = \cup A_\alpha$, where the A_α are connected components of A and if B is nonempty and $B \subset A$ such that $B \cap A_\alpha$ is compact and connected for each α then B is a set-valued retract of A .

Proof: Choose β such that $A_\beta \cap B \neq \phi$. Define

$$G(x) = \begin{cases} B \cap A_\alpha & \text{if} \quad x \in A_\alpha \quad \text{and} \quad B \cap A_\alpha \neq \phi \\ B \cap A_\beta & \text{if} \quad x \in A_\alpha \quad \text{and} \quad B \cap A_\alpha = \phi . \end{cases}$$

Then it is readily seen that G is the required map.

The next question is, do there exist any regions where $B \subset A$ is not a set-valued retract of A ? The following theorem answers this in the affirmative, indicating a wide class of examples.

Theorem 5.12: If A is connected and compact and if B is a set-valued retract of A , then B is connected and compact.

Proof: Let $G : A \to c(R^{n+1})$ be the associated set-valued mapping guaranteed by the definition. Then $G(x) \subset B$ for all $x \in A$ and if $x \in B$, $x \in G(x)$ so $G(A) = B$. But by the requirements on G and lemma 5.9, $G(A)$ is compact and connected, so B is also.

We now can generalize Wazewski's principle for generalized differential equations.

Theorem 5.13: If there exists a set $Z \subset \overline{V}$ such that $Z \cap \partial V$ is a set-valued retract of ∂V but not a set-valued retract of Z and if all consequent points are strict, then there exists a solution starting in Z which

does not leave V .

Proof: Assume for every $(t,x) \in Z$, every solution through (t,x) leaves V . Then the consequent map $C : Z \to c(\partial V)$ is u.s.c. on Z . Let $G : \partial V \to c(Z \cap \partial V)$ be the set-valued function guaranteed by the hypothesis that $Z \cap \partial V$ is a set-valued retract of ∂V .

Consider the composite function $GC : Z \to c(Z \cap \partial V)$. Using lemma 5.9, $G(C(t,x))$ is compact and connected for each $(t,x) \in Z$. If $(t,x) \in Z \cap \partial V$, $C(t,x) = \{(t,x)\}$, thus $(t,x) \in GC(t,x)$. If GC can be shown to be u.s.c. then this map will show that $Z \cap \partial V$ is a set-valued retract of Z which is a contradiction proving the theorem.

Claim: GC is u.s.c.

Proof of claim:

Given $\epsilon > 0$ and $(t_o,x_o) \in Z$, then for every $(t_\alpha,x_\alpha) \in C(t_o,x_o)$ choose δ_α such that if $d((t,x),(t_\alpha,x_\alpha)) < \delta_\alpha$ then $G(t,x) \subset S(G(t_\alpha,x_\alpha),\epsilon)$. $C(t_o,x_o)$ is compact so choose a finite number of $(t_{\alpha_i},x_{\alpha_i})$, $i = 1,\cdots,n$, such that

$$O \equiv \bigcup_{i=1}^{n} \{(t,x) : d((t,x),(t_{\alpha_i},x_{\alpha_i})) < \delta_{\alpha_i}\} \supset C(t_o,x_o) .$$

Let $\delta^* = \frac{1}{2}\,\text{dist}\,(C(t_o,x_o),P^{n+1} - O)$ and choose δ such that $d((t,x),(t_o,x_o)) < \delta$ implies $C(t,x) \subset S(C(t_o,x_o),\delta^*)$. Then if $d((t,x),(t_o,x_o)) < \delta$,

$$
\begin{aligned}
GC(t,x) &\subset G(S(C(t_o,x_o),\delta^*)) \\
&\subset G(O) \\
&\subset \bigcup_{i=1}^{n} G(\{(t,x) : d((t,x),(t_{\alpha_i},x_{\alpha_i})) < \delta_{\alpha_i}\}) \\
&\subset \bigcup_{i=1}^{n} S(G(t_{\alpha_i},x_{\alpha_i}),\epsilon) \\
&\subset S(\bigcup_{i=1}^{n} G(t_{\alpha_i},x_{\alpha_i}),\epsilon) \\
&\subset S(G(C(t_o,x_o),\epsilon)) \quad .
\end{aligned}
$$

Thus the theorem and claim are proven.

Remark: If $Z \subset \overline{V}$ is a compact, connected subset, if ∂V has more than one component all of which are uniformly bounded from each other, if all consequent points are strict and if Z intersected with each component of ∂V is connected, then the above theorem applies.

Let us return to the situation considered in Theorem 5.1 and impose the following strengthened hypotheses:

(HS') 1. $w(t,y) < z(t,y)$ for every $(t,y) \in R^n$.

2. w and z are continuous.

3. For every $t_o \in [a,\infty)$ and every $(x_1,\cdots,x_n) \in R^n$, if $x_1 = z(t_o,x_2,\cdots,x_n)$, then for every solution x to (P) with $x(t_o) = (x_1,\cdots,x_n)$, it follows that $x_1(t) > z(t,x_2(t),\cdots,x_n(t))$ for t on some right neighborhood of t_o where $x(t) = (x_1(t),\cdots,x_n(t))$.

4. For every $t_o \in [a,\infty)$ and every $(x_1,\cdots,x_n) \in R^n$, if $x_1 = w(t_o,x_2,\cdots,x_n)$ then for every solution x to (P) with $x(t_o) = (x_1,\cdots,x_n)$ it follows that $x_1(t) < w(t,x_2(t),\cdots,x_n(t))$ for t on some right neighborhood of t_o where $x(t) = (x_1(t),\cdots,x_n(t))$.

The following result follows immediately.

<u>Corollary 5.14</u>: Let S_1 be a pathwise connected set in $C(t_o)$ which intersects $W(t_o)$ and $Z(t_o)$ (Recall definition (N)). Let F satisfy (HO) and w and z satisfy (HS'). Then there exists a solution x to (P) with $x(t_o)$ in the interior of $C(t_o)$ such that $x(t)$ belongs to the interior of $C(t)$ for all t.

<u>Proof</u>: Let $w \in S_1 \cap W(t_o)$ and $z \in S_1 \cap Z(t_o)$. Then there exists a continuous function $f : [0,1] \to S_1$ such that $f(0) = w$ and $f(1) = z$. Define:

$$b = \sup\{s : f(s) \cap W(t_o) \neq \phi\}$$

$c = \inf\{s : f(s) \cap Z(t_o) \neq \phi$ and $f(r) \cap W(t_o) = \phi$ if $r \in [t,1]\}$. By continuity, $f(c) \in Z(t_o)$ and $f(b) \in W(t_o)$ and if $s \in (c,b)$, $f(s) \cap Z(t_o) = \phi$ and $f(s) \cap W(t_o) = \phi$. Hence $f([b,c])$ is a compact, connected subset of $C(t_o)$ such that

$$f([b,c]) \cap Z(t_o) = \{c\} \quad \text{and} \quad f([b,c]) \cap W(t_o) = \{b\} .$$

Therefore the result follows from theorem 5.11, theorem 5.12, and theorem 5.13 letting

$$V = \text{interior of } \{\{t\} \times C(t) : t \in [a,\infty)\} \quad \text{and}$$

$$Z = \{t_o\} \times f([b,c]) .$$

The third result constructs the set V as $V = \{x : v(x) \leq \lambda\}$ for an appropriate function v. This was motivated by Roxin [27, Lemma 9.4] and is related to the idea behind Lyapunov functions. For this case, it will be necessary to assume that $F : R^{n+1} \to cc(R^n)$ is continuous. A lemma must first be stated and proved.

Lemma 5.14: If $F : R^{n+1} \to cc(R^n)$ and if $x : R \to R^n$ is absolutely continuous such that $x'(t) \in F(t,x(t))$ a.e. then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < s < \delta$,

$$d_1(\frac{1}{s} x\ (t_o+s)\ ,\ \frac{x(t_o)}{s} + F(t_o,x(t_o))) < \epsilon\ .$$

Proof: Choose δ such that if $0 < s < \delta$ then

$$D(F(t_o,x(t_o)),F(t_o+s,x(t_o+s))) < \epsilon \quad \text{so}$$

$$d_1(x'(t_o+s),F(t_o,x(t_o))) < \epsilon \quad \text{a.e.}$$

Thus $x'(t_o+s) \in S(F(t_o,x(t_o)),\epsilon)$ a.e. so since $S(F(t_o,x(t_o)),\epsilon)$ is a compact, convex set,

$$\frac{1}{s}\int_{t_o}^{t_o+s} x'(r)\ dr \in S(F(t_o,x(t_o)),\epsilon)$$

for $0 < s < \delta$. Since x is absolutely continuous,

$$x(t_o+s) - x(t_o) = \int_{t_o}^{t_o+s} x'(r)\ dr\ ,$$

hence

$$\frac{1}{s}(x(t_o+s) - x(t_o)) \in S(F(t_o,x(t_o)),\epsilon) \quad ,$$

from which the lemma follows.

Notation: (1) For a l.s.c. function $v : R^n \to R$ such that $h(y) = \lim_{r\to 0^+} (\frac{1}{r} v(ry))$ exists for all y, then define

$$w(r,y) = \begin{cases} \frac{1}{r} v(ry) & r \neq 0 \\ h(y) & r = 0 \quad . \end{cases}$$

(2) $D^*(t,x) = \limsup_{s\to 0^+} [\sup\{ \frac{v(y) - v(x)}{s} : y \in x + sF(t,x)\}$.

(3) $A(\lambda) = \{x : v(x) \leq \lambda\}$.

With the above notation, the following theorem can now be stated and proved.

<u>Theorem 5.15</u>: If $D^*(x,t) \leq 0$ for all x and t, if $w(r,y)$ is uniformly continuous in a neighborhood of $r = 0$ and all y, and if $x_o \in A(\lambda)$, and if x is a solution to (P) such that $x(t_o) = x_o$, then $x(t) \in A(\lambda)$ for all $t \geq t_o$.

<u>Proof</u>:

Claim: For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < s < \delta$ then $d(\frac{v(x(t+s))}{s}, \frac{v(x+sF(t,x(t)))}{s}) < \epsilon$.

Proof of claim:

Choose δ_1 and δ_2 such that if $0 < r < \delta_1$ and $d(y_1,y_2) < \delta_2$.

Then $d(w(r,y_1),w(r,y_2)) < \epsilon$. By lemma 5.14 for some δ , $0 < \delta < \delta_1$, if $0 < s < \delta$ then

$$d_1(\frac{x(t+s)}{s} , \frac{x(t) + sF(t,x(t))}{s}) < \delta_2$$

hence

$$d(w(s, \frac{x(t+s)}{s}) , w(s, \frac{x(t) + sF(t,x(t))}{s})) < \epsilon$$

hence

$$d(\frac{v(x(t+s))}{s} , \frac{v(x + sF(t,x(t)))}{s}) < \epsilon ,$$

which proves the claim.

Hence, if $0 < s < \delta$,

$$\frac{v(x(t+s)) - v(x(t))}{s} \leq \sup \{\frac{v(y) - v(x(t))}{s} : y \in x + s\ F(t,x(t))\} + \epsilon .$$

Thus,

$$\begin{aligned} D^+v(x(t)) &\equiv \limsup_{s\to 0^+} [\frac{v(x(t+s)) - v(x(t))}{s}] \\ &\leq \limsup_{s\to 0^+} [\sup \{\frac{v(y) - v(x(t))}{s} : y \in x+sF(t,x)\}] + \epsilon \\ &\leq D(x,t) + \epsilon \leq \epsilon . \end{aligned}$$

This is true for any $\epsilon > 0$, so $D^+v(x(t)) \leq 0$. But $v(x(\cdot))$ is l.s.c., hence $v(x(\cdot))$ is monotonically decreasing [27, lemma 9.3]. Thus $v(x(t)) \leq v(x(t_o)) \leq \lambda$ for $t \geq t_o$ hence $v(x(t)) \in A(\lambda)$ for $t \geq t_o$.

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